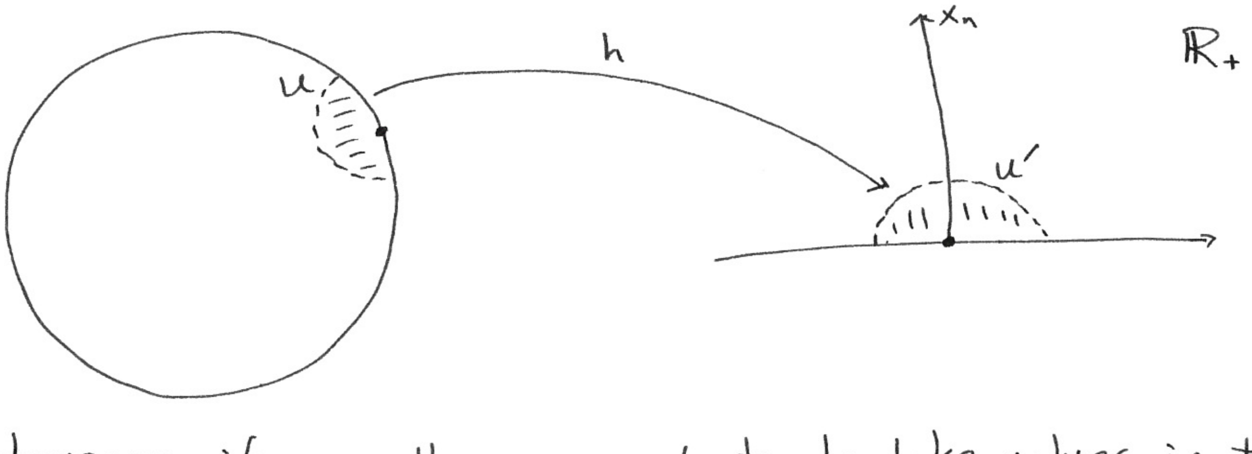
Manifolds with boundary

The disk $\overline{D_r(0)} = \{x \in \mathbb{R}^n \mid |x| \leq r\}$ fails to be a manifold because points x with $|x| = r$ do not have a nhood in $\overline{D_r(0)}$ diffeomorphic to an open set in $\mathbb{R}^n$.



However, if we allow our charts to take values in the

$$h : \mathcal{U} \longrightarrow \mathcal{U}' \subseteq \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$$

half space $\{x_n \geq 0\}$, the hyperplane $\{x_n = 0\} \cong \mathbb{R}^{n-1}$ exactly models the "boundary points" of $\overline{D_r(0)}$. Write $\partial\mathbb{R}^n_+ = \{x_n = 0\}$.

Def: A smooth n-manifold with boundary is a second countable Hausdorff space with an atlas

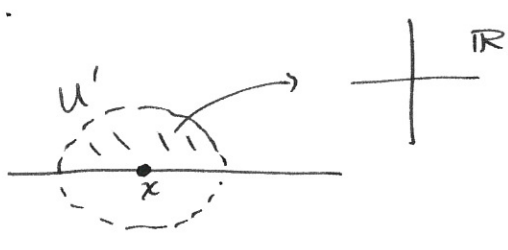
$$\{h_\alpha : \mathcal{U} \xrightarrow{\cong} \mathcal{U}'_\alpha\}$$

where each $\mathcal{U}'_\alpha \subset \mathbb{R}^n_+$ is open and the transition functions

$$h_{\alpha\beta} = h_\beta h_\alpha^{-1}$$

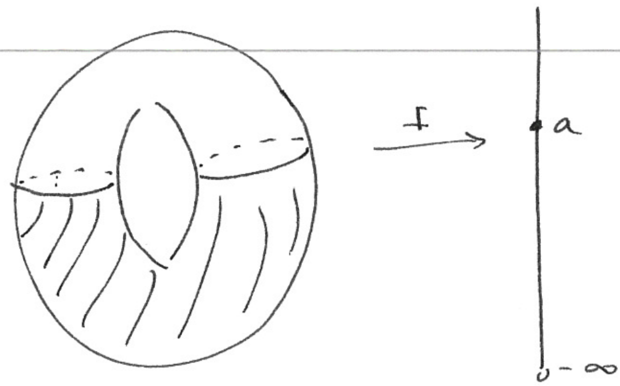
are C^∞.

Recall: a function $\mathcal{U}' \longrightarrow \mathbb{R}^n_+$ is smooth (C^∞) at $x \in \mathcal{U}' \cap \partial\mathbb{R}^n_+$ if it has an extension to a nhood of x in $\mathbb{R}^n$ which is smooth at x.



The boundary of M, ∂M, is the set of $x \in M$ which map to $\partial\mathbb{R}^n_+$ under one (and hence all) charts. The interior of M is $M - \partial M$.

Examples: If M is an ordinary manifold (i.e. w/o boundary) and $f: M \to \mathbb{R}$ has $a \in \mathbb{R}$ as a regular value, then $f^{-1}(-\infty, a]$ is a manifold with boundary, and $\partial f^{-1}(-\infty, a] = f^{-1}(a)$.



Prop: If $(M, \partial M)$ is an n-manifold with boundary, then $M - \partial M$ is an n-manifold and ∂M an $(n-1)$-manifold.

Proof: For ∂M we use the restrictions of charts

$$\begin{array}{ccccc} U & \xrightarrow{h} & U' & \subseteq & \mathbb{R}^n_+ \\ \cup & & \cup & & \cup \\ U \cap \partial M & \xrightarrow{h} & U' \cap \partial \mathbb{R}^n_+ & \subseteq & \mathbb{R}^{n-1} \end{array}$$

Remark: Connected 1-manifolds are homeomorphic to one of

1) $\mathbb{R} \cong (a,b)$
2) $(-\infty, a] \cong [b,\infty)$
3) $[0,1] \cong [a,b]$

or 4) S^1

If it is also compact then only (3) and (4) occur. A compact 1-manifold is therefore a disjoint union of closed intervals and circles.

Cor: If M is a compact 1-manifold then $\#\partial M$ is even.

This will allow us to prove the Brouwer Fixed Point Theorem shortly. First, we need

Theorem: There is no smooth retraction $\overline{D^n} \xrightarrow[r]{f} \partial\overline{D^n} \cong S^{n-1}$.

Proof: Recall that r is a retraction if $r|\partial\overline{D^n}$ is the identity. If we had such an r then we could find a regular value $x \in S^{n-1}$ and consider $r^{-1}(x)$. This is a compact submanifold with boundary of $\overline{D^n}$, and

$$\partial r^{-1}(x) = r^{-1}(x) \cap \partial\overline{D^n}.$$

Now $y \in \partial r^{-1}(x) \Rightarrow y \in \partial\overline{D^n} \Rightarrow r(y) = y$, since r is a retraction, while $y \in \partial r^{-1}(x) \Rightarrow y \in r^{-1}(x) \Rightarrow r(y) = x$. Thus $\#\partial r^{-1}(x) = 1$. But the Corollary above tells us that $\#\partial r^{-1}(x)$ is even. This contradiction implies that no such r exists. //

Theorem: There is no continuous retraction $r: \overline{D^n} \to \partial\overline{D^n}$.

Proof: Given such a retraction, we would get a retraction $R: \overline{D_2^n} \to \partial\overline{D_2^n}$ by composing

$$x \longmapsto \begin{Bmatrix} r(x) & |x| \le 1 \\ x & 1 \le |x| \le 2 \end{Bmatrix} \quad \text{with} \quad x \longmapsto 2\frac{x}{|x|}.$$

This is smooth in a neighborhood of $\partial\overline{D_2^n}$, so there is a smooth retraction $\overline{D_2^n} \to \partial\overline{D_2^n}$. The previous theorem shows this cannot happen. //