Naturality

If $f : M \longrightarrow N$ then $Tf : TM \longrightarrow TN$ induces

$$T^{k,0}f : T^k M \longrightarrow T^k N$$ and dually

~~and~~ $$T^{0,k}f : T^{0,k} N \longrightarrow T^{0,k} M$$

and $$\Lambda^k f : \Lambda^k N \longrightarrow \Lambda^k M$$

or $$f^* : \Lambda N \longrightarrow \Lambda M$$

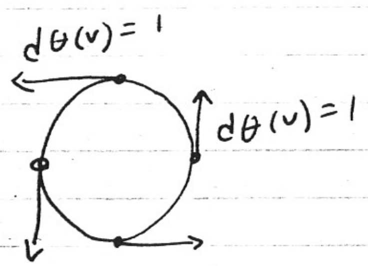
where $f^*(\omega)(v_1, \dots, v_r) = \omega(Tf(v_1), \dots, Tf(v_r))$.

Examples: $\Lambda(\mathbb{R}^2)$ is spanned by dx, dy

$$\Big\downarrow i^*$$

$\Lambda(S^1)$ is spaned by $d\theta = x\,dy - y\,dx$

$S^1 \xrightarrow[i]{} \mathbb{R}^2$



$i^* dx = ?$

$$\begin{tikzcd}
\Lambda(M) & \Lambda(N) \arrow[l, "f^*"'] \\
M \arrow[u, bend right, "f^*_\omega"'] \arrow[r, "f"] & N \arrow[u, bend right, "\omega"']
\end{tikzcd}$$

$d\theta_{(x,y)}(u,v) = xv - yu$

(Note $xu + yv = 0$)

$i^* dx_{(x,y)}(u,v) = u$

Now $u = -yv/x$

$v = -xu/y$

How do we express this in terms of $d\theta$

$d\theta_{(x,y)}(-y, +x) = 1$

$i^* dx_{(x,y)}(-y, x) = -y$

so $i^* dx = -y\,d\theta$

Check $-y\,d\theta_{(x,y)}(u,v) = -yxv + y^2u = -yx\left(\frac{-xu}{y}\right) + y^2u = (x^2+y^2)u$.
$= u$.

Similarly $i^*dy_{(x,y)}(-y,x) = x$ so $i^*dy = x\,d\theta$

Check $x\,d\theta_{(x,y)}(\ ,v) = x^2v - xyu = x^2v + y^2v = v. = i^*dy_{(x,y)}(u,v)$. //

Note $x^2+y^2 = 1$ so $2x\,dx + 2y\,dy = 0$

or $x\,dx + y\,dy = 0$

i.e $x(-y\,d\theta) + y(x\,d\theta) = (-xy+xy)\,d\theta = 0$. OK.

<u>Contemplate</u>: $\omega = dx\wedge dy + dy\wedge dz + dz\wedge dx$ on $S^2 \subset \mathbb{R}^2$

At $\begin{pmatrix} a\\ b\\ c\end{pmatrix}$ we have tangents $\begin{pmatrix} b\\ -a\\ 0\end{pmatrix}$, $\begin{pmatrix} c\\ 0\\ -a\end{pmatrix}$ and $\begin{pmatrix} 0\\ c\\ -b\end{pmatrix}$

And $\omega\left(\begin{pmatrix} b\\ -a\\ 0\end{pmatrix}, \begin{pmatrix} c\\ 0\\ -a\end{pmatrix}\right) = (b\cdot 0 - (-a)c) + (-a(-a)) + (0 - b(-a))$

$= ac + a^2 + ab = a(a+b+c)$

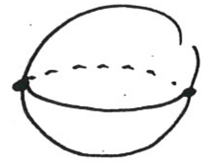
reverse

add − after reversal

$\omega\left(\begin{pmatrix} b\\ -a\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ c\\ -b\end{pmatrix}\right) = bc - 0 + ab - 0 + b^2 = b(a+b+c)$

$\omega\left(\begin{pmatrix} c\\ 0\\ -a\end{pmatrix}, \begin{pmatrix} 0\\ c\\ -b\end{pmatrix}\right) = c^2 + 0 - (-ac) + 0 - (-bc) = c(a+b+c)$

Therefore ω vanishes along the great circle $a+b+c=0$ in S^2. Try again



$$\omega = x\, dy\wedge dz + y\, dz\wedge dx + z\, dx\wedge dy$$

$$\omega\left(\begin{pmatrix} b\\-a\\0\end{pmatrix}, \begin{pmatrix} c\\0\\-a\end{pmatrix}\right) = a(a^2) + b(ab) + c(ac)$$

$$= a(a^2+b^2+c^2) = a$$

$$\omega\left(\begin{pmatrix} c\\0\\-a\end{pmatrix}, \begin{pmatrix} 0\\c\\-b\end{pmatrix}\right) = a(0+ac) + b(bc) + c(c^2)$$

$$= c(a^2+b^2+c^2) = c$$

$$\omega\left(\begin{pmatrix} 0\\c\\-b\end{pmatrix}, \begin{pmatrix} b\\-a\\0\end{pmatrix}\right) = a(-ab) + b(-b^2) + c(-bc)$$

$$= b(-a^2-b^2-c^2) = -b.$$

Prop: $f: M \to N$ smooth $\Rightarrow$

1. $f^*: \Gamma(\Lambda(N)) \to \Gamma(\Lambda(M))$ is an alg hom
2. $f^* d\omega = d f^*\omega$
3. $f^*\omega(V_1,\dots,V_r) = \omega(Tf(V_1),\dots,Tf(V_r))$

   for v.f. $V_i \in \Gamma(TM)$.