Exercises #4 due Monday Feb 22, 1999.

1. Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^t = I\}$ is compact.

2. For which real $a \geq 0$ does the hyperboloid $x^2+y^2-z^2=1$ intersect the sphere $x^2+y^2+z^2=a^2$ transversally. Describe the intersection for each $a \geq 0$.

3. Show that $\mathbb{R}P^n \times \mathbb{R}P^m \longrightarrow \mathbb{R}P^{mn+m+n}$ by

$$([x_0,\dots,x_n],[y_0,\dots,y_m]) \longmapsto [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$$

is an embedding. (It is called the Segré embedding.)

4. Let $P \subset SO(3)$ be $P = \{Q \mid Q = Q^t\} \setminus \{I\}$.

(a) Show that P is a 2 dimensional compact submanifold of $SO(3)$.

(b) Let $f: \mathbb{R}P^2 \to P$ send the line $\ell \in \mathbb{R}P^2$ to rotation through $180°$ about ℓ. Show that f is a diffeomorphism $\mathbb{R}P^2 \to P$.

5. Show that any product of spheres can be embedded in Euclidean space of one dimension higher.

Hints: (1) Show that $S^n \times \mathbb{R}$ can be embedded into $\mathbb{R}^{n+1}$! (The "fat" sphere)

(2) Explicitly describe, using trigonometric functions, an embedding $T = S^1 \times S^1 \to \mathbb{R}^3$.