Def: A topological manifold of dimension n is a 2^{nd} countable, Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

Note: Some people do not require 2^{nd} countable, i.e. having a countable basis for the topology, in their definition of a manifold. This would allow the long line [Munkres, p. 159], which, by our definition, is not a manifold.

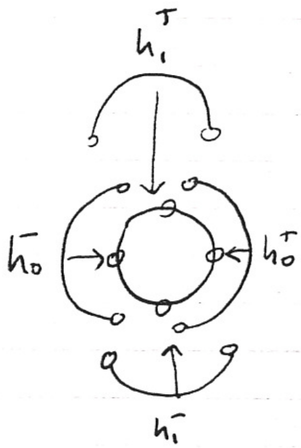
Examples: 0) ϕ is an n-manifold for any $n \geq 0$. (we will see that a non-empty manifold has a unique dimension.)

1) Any open set in $\mathbb{R}^n$ is an n-manifold (i.e. topological manifold of dimension n.)

2) The unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ is an n-manifold. To show it is locally Euclidean, let

$$D^n = \{x \in \mathbb{R}^n \mid (x,x) < 1\}$$

be the open n-disk, and let



$$h_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \sqrt{1-x_1^2-\cdots-x_n^2} \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix}$$

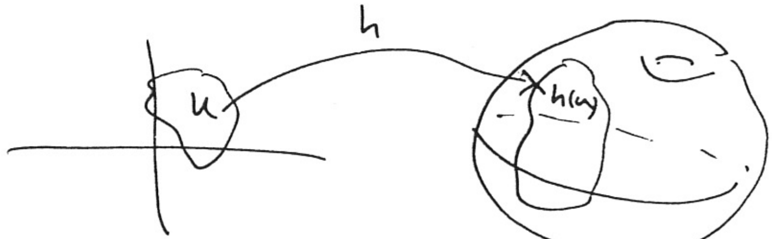
for $i = 0, \ldots, n$. This is a homeomorphism $D^n \xrightarrow{h_i} U_i \subseteq S^n$ onto an open set U_i in S^n. We also let

$$h_i^+ = h_i \text{ and } h_i^- = -h_i = Th_i,$$

where $Tx = -x$ is the antipodal map. Then $\{h_0^\pm, \ldots, h_n^\pm\}$ show that every point in S^n has a nbhd homeomorphic to $\mathbb{R}^n$.

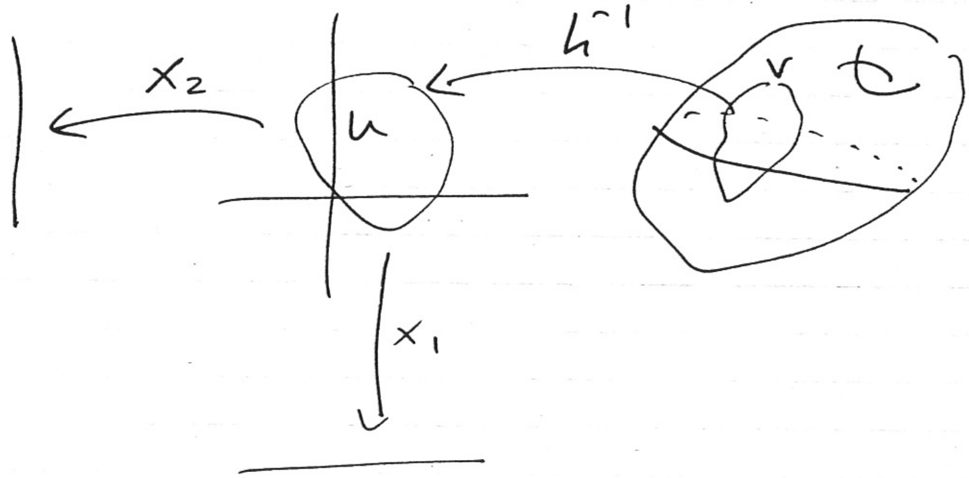
<u>Def</u>: If $U \subseteq \mathbb{R}^n$ is an open set and M is a topological space then a homeomorphism $h : U \to h(U) \subseteq M$ to an open set $h(U)$ in M is called a <u>chart</u>.

$$\begin{array}{ccc} U & \longrightarrow & h(U) \\ \cap & & \cap \\ \mathbb{R}^n & & M \end{array}$$



An <u>atlas</u> $\mathcal{h}$ on a topological manifold M is a collection $\mathcal{h} = \{h_\alpha\}$ of charts which cover M; $M = \bigcup_\alpha \operatorname{Im}(h_\alpha)$.

If $h : U \xrightarrow{\cong} V \subseteq M$ is a chart, its inverse $h^{-1} : V \to U \subseteq \mathbb{R}^n$ is called <u>a coordinate system</u> on V. The composites $x_i h^{-1} : V \to \mathbb{R}^n \to \mathbb{R}$ with the i^{th} coordinate functions $x_i : \mathbb{R}^n \to \mathbb{R}$ are called <u>coordinate functions on V</u>.



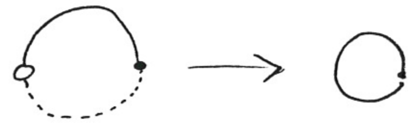
<u>Note</u>: Our atlas for S^n used $2n+2$ charts, but we need only two: stereographic projection from any two distinct points.

This is the minimum possible (Exercise 4).

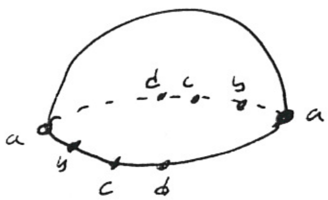
Example 3: $\mathbb{R}P^n$ - real projective n-space, is the quotient $S^n/\sim$ where $\sim$ is the equivalence relation $\alpha \sim \beta$ iff $\alpha = \pm\beta$. Let $p_n : S^n \to \mathbb{R}P^n$ be the quotient map.

$n=0$ $\quad S^0 = \{\pm 1\} \xrightarrow{p_0} \mathbb{R}P^0 = \text{point}$

$n=1$ $\quad S^1 \xrightarrow{p_1} \mathbb{R}P^1 \cong S^1$ is of degree 2



$n=2$ $\quad p_2$ restricted to upper hemisphere covers $\mathbb{R}P^2$ so we can construct $\mathbb{R}P^2$ from the closed disk or upper hemisphere by identifying antipodal points on the boundary



or, cut a disk out of a sphere and sew a Möbius band back in.

Our first atlas for S^n provides one for $\mathbb{R}P^n$. Let $k_i = p_n h_i = p_n h_i$. On $im(h_i)$, p is a homeomorphism, so k_i is a homeomorphism

$$\begin{array}{ccc} D^n & \longrightarrow & U_i \subseteq \mathbb{R}P^n \\ \| & & \| \\ \{x \in \mathbb{R}^n \mid (x,x) < 1\} & & \{[x_0,\dots,x_n] \mid x_i \neq 0\} \end{array}$$

where $[x_0,\dots,x_n]$ is the equivalence class of $(x_0,\dots,x_n) \in S^n$, that is $[x_0,\dots,x_n] = \{(x_0,\dots,x_n), (-x_0,-x_1,\dots,-x_n)\}$.

Aside:
In fact this is the most efficient atlas possible. To prove this we need a bit of algebraic topology

Thm: If the cohomology ring of X has a product of n elements which is nonzero, then at least $n+1$ simply conn charts are needed to cover X.

Example 4: Another view of $\mathbb{RP}^n$. Note that each point of $\mathbb{RP}^n$, $[x]=\{x,-x\}$ determines a line through the origin in $\mathbb{R}^{n+1}$, namely, the line $\{rx \mid r \in \mathbb{R}\}$. From this point of view, we define an equivalence relation $\approx$ on $\mathbb{R}^{n+1}-0$ by

$$x \approx y \quad \text{iff} \quad x = ry \quad \text{for some } r \in \mathbb{R}$$
(necessarily non zero)

Then $\mathbb{RP}^n = S^n/\sim \;\cong\; \mathbb{R}^{n+1}-0/\approx$. Given $(x_0,\dots,x_n) \in \mathbb{R}^{n+1}-0$ let $[x_0,\dots,x_n]$ be the equivalence class of $(x_0,\dots,x_n)$. We call these the homogeneous coordinates of the point. Of course,

$$[x_0,\dots,x_n] = [rx_0,\dots,rx_n] \quad \text{for any } r \neq 0.$$

If $p(x_0,\dots,x_n)$ is any homogeneous polynomial, e.g. $x_0x_1 - x_2^2$, the set

$$V(p) = \{[x_0,\dots,x_n] \mid p(x_0,\dots,x_n) = 0\}$$

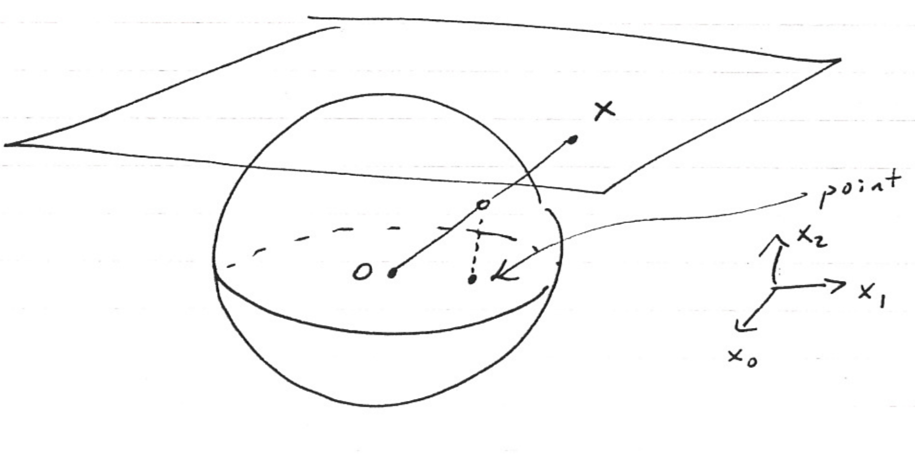
is well defined.

Let $U_i \subset \mathbb{RP}^n$ be $U_i = \{[x_0,\dots,x_n] \mid x_i \neq 0\}$. There is a chart

$$l_i : \mathbb{R}^n \longrightarrow U_i$$

by $l_i(x_1,\dots,x_n) = [x_1,\dots,x_{i-1}, 1, x_i,\dots,x_n]$

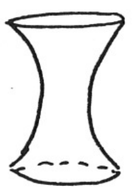
"ith" coordinate (starting with 0)



point in D^2 such that k_2 of it equals $l_2(x)$.

<u>One more example</u>: The surface $x^2+y^2-z^2=a$ is a manifold if $a \neq 0$. There are two cases.

<u>Case 1</u>: $x^2+y^2=z^2+a$, $a>0$. Two charts suffice:



$$h_1 : (-\pi,\pi)\times \mathbb{R} \longrightarrow M$$

$$h_2 : (0, 2\pi)\times \mathbb{R} \longrightarrow M$$

by $\quad h_i(\theta, z) = \left(\sqrt{z^2+a}\,\cos\theta,\ \sqrt{z^2+a}\,\sin\theta,\ z\right)$

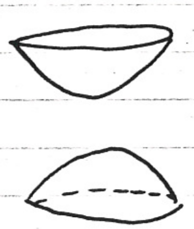
These are both homeomorphisms to their images. For example,

$$h_1^{-1} : im(h_1) \longrightarrow (-\pi,\pi)\times \mathbb{R}$$

is $\quad h_1^{-1}(x,y,z) = \left(\mathrm{Tan}^{-1}(x,y),\ z\right)$, where

$$\mathrm{Tan}^{-1}(x,y) = \begin{cases} \tan^{-1}(y/x) & x>0 \\ \cot^{-1}(x/y) & y>0 \\ -\pi + \cot^{-1}(x/y) & y<0 \end{cases}$$

<u>Case 2</u>: $x^2+y^2+a=z^2$, $a>0$. Again two patches (charts) are sufficient.



$$h_+ : \mathbb{R}^2 \longrightarrow M, \quad h_+(x,y) = \left(x, y, \sqrt{x^2+y^2+a}\right)$$

and $\quad h_- : \mathbb{R}^2 \longrightarrow M, \quad h_-(x,y) = \left(x,y,-\sqrt{x^2+y^2+a}\right)$

Inverses are $\quad h_\pm^{-1}(x,y,z) = (x,y)$.