Lie Groups

Lie groups are named in honor of Sophus Lie, who carried out their first systematic study in the late 1900's.

Def: A **Lie group** is a smooth manifold G which is also a group, such that multiplication $m: G\times G \to G$ and inverse $r: G \to G$ are smooth functions.

Examples: (1) Any discrete group is a 0-dimensional Lie group.

(2) The general linear groups over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are

$$Gl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}$$

and similarly for $Gl_n(\mathbb{C})$ and $Gl_n(\mathbb{H})$.

(3) The special linear groups $Sl_n(\mathbb{R})$ and $Sl_n(\mathbb{C})$ are the subgroups of $Gl_n(\mathbb{R})$ and $Gl_n(\mathbb{C})$ of determinant 1.

(4) The classical compact Lie groups

$O(n) = \{A \in Gl_n(\mathbb{R}) \mid AA^t = I\}$, the orthogonal group
$SO(n) = O(n) \cap Sl_n(\mathbb{R})$, the special orthogonal group

$U(n) = \{A \in Gl_n(\mathbb{C}) \mid A\bar{A}^t = I\}$, the unitary group
$SU(n) = U(n) \cap Sl_n(\mathbb{C})$, the special unitary group

$Sp(n) = \{A \in Gl_n(\mathbb{H}) \mid A\bar{A}^t = I\}$, the symplectic group

(Note the conjugate of a quaternion $q = a+bi+cj+dk$ is $\bar{q} = a-bi-cj-dk$ and $q\bar{q} = \bar{q}q = \|q\|^2 = a^2+b^2+c^2+d^2$.)

(Note: $O(1) = \mathbb{Z}/2\mathbb{Z}$, $SO(2) = S^1 = U(1)$, $SU(2) = S^3 = Sp(1)$.

Def: $l_g : G \to G$, left translation by $g \in G$ is $l_g(h) = gh$,

and $r_g : G \to G$, right translation by $g \in G$ is $r_g(h) = hg$.

<u>Prop</u>: l_g and r_g are diffeomorphisms.

<u>Pf</u>: They are smooth since they are restrictions of the smooth map $m: G\times G \to G$. They are diffeomorphisms since $(l_g)^{-1} = l_{g^{-1}}$ and $(r_g)^{-1} = r_{g^{-1}}$. //

<u>Def</u>: A vector field v on G is <u>left invariant</u> if

$$Tl_g \circ v = v \circ l_g$$

for every $g \in G$.

$$\begin{tikzcd} TG \arrow[r, "Tl_g"] & TG \\ G \arrow[u, "v"] \arrow[r, "l_g"] & G \arrow[u, "v"'] \end{tikzcd}$$

Since a left invariant vector field v satisfies $v(g) = Tl_g(v(e))$, the entire vector field is determined by the single vector $v(e) \in T_eG$, $e \in G$ the identity element. Conversely, given any $A \in T_eG$ we let v^A be the vector field $v^A(g) = Tl_g(v^A(e)) = Tl_g(A)$. It is easy to check that this is a left invariant vector field, and that this gives a 1-1 correspondence

$$T_eG \longleftrightarrow \text{Left invariant vector fields on } G.$$

<u>Def</u>: The <u>Lie algebra of G</u> is $T_eG = \mathfrak{g}$

<u>Note</u>: This is a vector space, of course. It also has a <u>Lie bracket</u>

$$[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying (1) $[-,-]$ is bilinear

(2) $[Y,X] = -[X,Y]$

(3) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

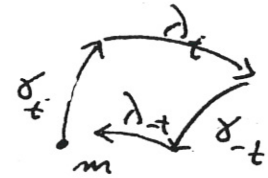
(the Jacobi identity)

This is a special case of a more general definition of the Lie bracket $[X,Y]$ of any two (smooth) vector fields X and Y on a manifold M. There are two ways to define $[X,Y]$:

(1) Consider curves $\gamma(t)$ and $\lambda(t)$ tangent to X and Y in the sense that

$$\gamma'(t) = X(\gamma(t)) \quad \text{and} \quad \lambda'(t) = Y(\lambda(t))$$

for each t. Let γ_t be $\gamma|[0,t]$ and let γ_{-t} be γ run backwards for time t, and similarly for λ.



Then

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}\left(\lambda_{-t}\circ\gamma_{-t}\circ\lambda_t\circ\gamma_t(m) - m\right)$$

(2) A tangent vector field X acts on C^∞ functions $M \xrightarrow{f} \mathbb{R}$ by

$$X(f)(m) = Df(m)(X(m)).$$

It is easy to check that $X(fg) = fX(g) + gX(f)$. We say that X is a <u>derivation</u> when it behaves this way on products. It is not hard to show that every derivation on $C^\infty(M)$ comes from a vector field. Then $[X,Y]$ is the vector field which acts by

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

In brief, $[X,Y] = XY - YX$. This formula works in any algebra to define an operation $[X,Y]$ satisfying properties (1) – (3). For example, in $M_n(\mathbb{R}^2)$, where we have subsets closed under $[-,-]$ such as

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \operatorname{tr}(A) = 0\} = T_I\, Sl_n(\mathbb{R}), \text{ and}$$

$$\mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) \mid A + A^t = 0\} = T_I\, O(n).$$

<u>Theorem</u>: A Lie group is parallelizable.

<u>Proof</u>: Define a diffeomorphism

$$\begin{array}{ccc} G \times T_eG & \xrightarrow{t} & TG \\ \pi_1 \downarrow & & \downarrow \tau \\ G & = & G \end{array}$$

by $t(g, A) = v^A(g) = Tl_g(A)$. The inverse, $t^{-1}(v) = (\tau v, Tl_{\tau(v)^{-1}}(v))$ shows that t is one-to-one and onto. To show that it is a smooth map, consider

$$G \times T_eG \xrightarrow{s \times i} TG \times TG \xrightarrow{Tm} TG$$

where $s(g) = 0_g$ is the zero-section, and i is the inclusion. Now, in general,

$$Tm_{(a,b)}(u, v) = T_a(r_b)(x) + T_b(l_a)(y)$$

by decomposing Dm into its components with respect to the two factors of $G \times G$. Therefore

$$Tm_{(g,e)}(0_g, A_e) = T_g(r_e)(0_g) + T_e(l_g)(A_g)$$

$$= T_e(l_g)(A_g)$$

$$= v^A(g)$$

$$= t(g, A_e)$$

where we use subscripts to indicate which tangent space an element belongs to: $0_g \in T_gG$ and $A_e \in T_eG$. Therefore, $t = Tm \circ (s \times i)$ is smooth since Tm and $s \times i$ are smooth. Similarly, we can factor the map $v_g \longmapsto Tl_{g^{-1}}(v)$ as the composite

$$TG \xrightarrow{\binom{\tau}{1}} G \times TG \xrightarrow{r \times 1} G \times TG \xrightarrow{s \times 1} TG \times TG \xrightarrow{Tm} TG$$

since $Tm_{(g^{-1}, g)}(0_{g^{-1}}, v) = 0 + T_g l_{g^{-1}}(v)$, showing t^{-1} is also smooth. //

<u>Examples</u>: 1) Once you have shown $\mathbb{RP}^2$ is not parallelizable, you will have a proof that $\mathbb{RP}^2$ does not support a group structure.

2) One can check that $\mathbb{RP}^3 = SO(3)$, and is therefore parallelizable, as follows. We can view S^3 as the unit quaternions

$$S^3 = \{a + bi + cj + dk \in \mathbb{H} \mid a^2+b^2+c^2+d^2 = 1\}$$

and $\mathbb{R}^3$ as the "pure quaternions" (real part 0)

$$\mathbb{R}^3 = \{bi + cj + dk \in \mathbb{H}\}.$$

Let S^3 act upon $\mathbb{R}^3$ by conjugation:

$$\begin{aligned} (g, x) &\longmapsto g x g^{-1} \\ S^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3. \end{aligned}$$

This is linear in x, and you can easily check that it defines an element of $SO(3)$, so that we have a group homomorphism

$$S^3 \longrightarrow SO(3).$$

This is onto, and its kernel is $\{\pm 1\}$, hence defines a diffeomorphism

$$\mathbb{RP}^3 \xrightarrow{\cong} SO(3).$$

Let us look at these ideas in a concrete example, $G = SO(3)$. We have seen that

$$T_e G = \{A \in M_3(\mathbb{R}) \mid A + A^t = 0\}$$
$$= \left\{ \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix} \middle| \ a, b, c \in \mathbb{R} \right\}$$

the skew-symmetric matrices. This is spanned by

$$e_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

which are the tangent vectors at 0 for the curves $\gamma_{ij} : \mathbb{R} \to SO(3)$

$$\gamma_{12}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_{13}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \text{ and}$$

$$\gamma_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

That is,

$$\gamma_{12}(0) = \gamma_{13}(0) = \gamma_{23}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e \in G$$

and

$$\gamma_{12}'(0) = e_{12}, \quad \gamma_{13}'(0) = e_{13}, \quad \gamma_{23}'(0) = e_{23}.$$

Clearly $\gamma_{ij}(\theta)$ represents rotation in the ij-plane by angle θ so we can think of three tangent directions at the identity as corresponding to rotations in the three coordinate planes.

Now, suppose $A \in SO(3)$. Since left translation by A is a linear function $M_3(\mathbb{R}) \to M_3(\mathbb{R})$, $Dl_A(e)(v) = A$, so the three matrices Ae_{12}, Ae_{13}, Ae_{23} span the tangent space to $SO(3)$ at A. More computationally, we can consider the three curves $\theta \mapsto A\gamma_{ij}(\theta)$. They pass through A when $\theta = 0$ and $\gamma_{ij}'(0) = Ae_{ij}$. They may be visualized as follows. If the columns of A are a_1, a_2 and $a_3 \in \mathbb{R}^3$ then (a_1, a_2, a_3) is a right handed orthonormal basis for $\mathbb{R}^3$, and Ae_{ij} is tangent to rotation in the plane spanned by a_i and a_j :