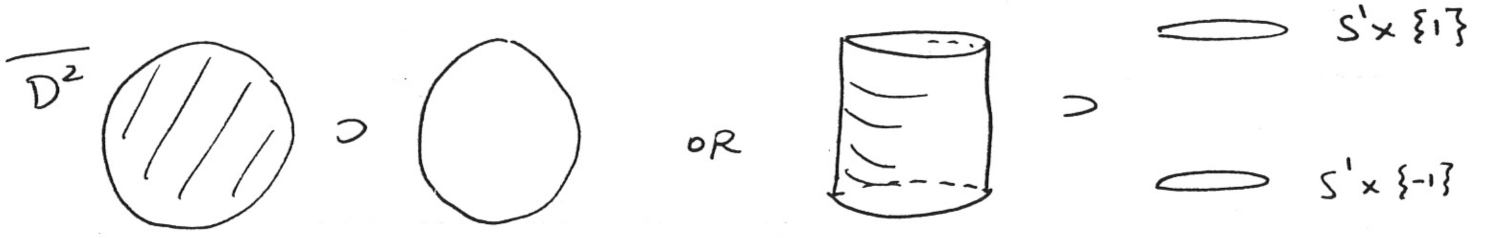
Manifolds with Boundary

Many natural spaces, like the closed n-disk $\overline{D^n} = \{x \in \mathbb{R}^n \mid |x| \le 1\}$, or the cylinder $\{(x,y,z) \mid x^2+y^2=1, -1 \le z \le 1\} = S^1 \times [-1,1]$, are not manifolds by the definition we have used so far, because of the presence of a <u>boundary</u>. A slight generalization will

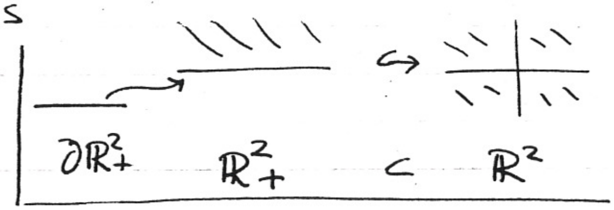


$$\overline{D^2} \supset \partial \overline{D^2} = S^1 \qquad S^1 \times [-1,1] \supset \partial(S^1 \times [-1,1])$$

allow us to study these objects as well. A powerful geometric theory called <u>cobordism theory</u> emerges from the relation between manifolds and their boundaries.

<u>Def</u>: An <u>n-dimensional half space</u> is

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n \ge 0\}$$



where $x = (x_1, \dots, x_n)$. Its <u>boundary</u> $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid x_n = 0\} \cong \mathbb{R}^{n-1}$ and <u>interior</u> $\mathrm{Int}(\mathbb{R}^n_+) = \{x \in \mathbb{R}^n_+ \mid x_n > 0\}$.

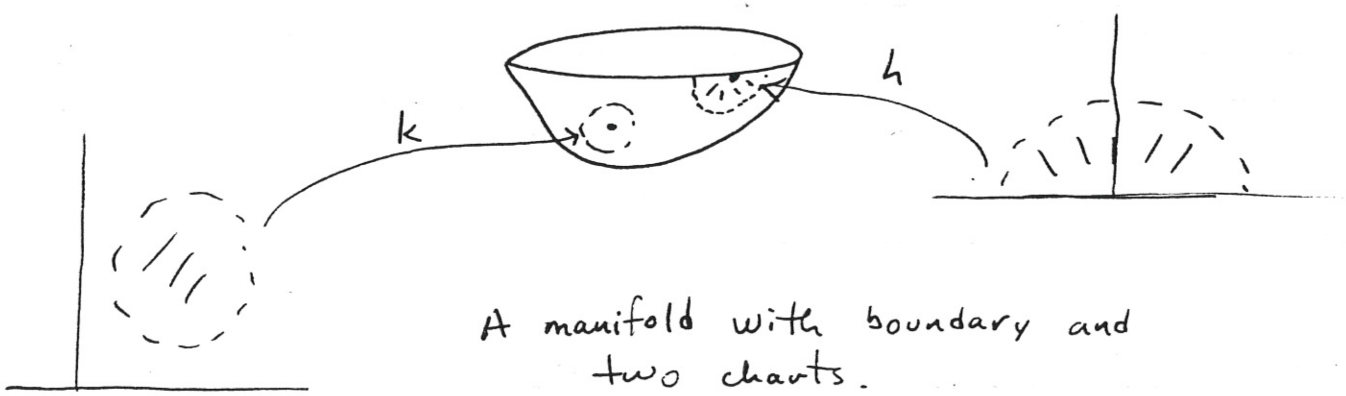
Of couse, we obtain the same space up to linear change of coordinates from $\{x \in \mathbb{R}^n \mid \ell(x) \ge 0\}$ for any affine function $\ell(x) = x_0 + Ax$.

The half space serves as our local model for our generaligation.

<u>Def</u>: An <u>m-manifold with boundary</u> M is a second countable Hausdorff space such that each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^m_+$. The <u>boundary</u> of M,

$$\partial M = \{x \in M \mid x \in h(\partial \mathbb{R}^m_+) \text{ for some chart } h\}, \quad \text{and}$$

the <u>interior of M</u>, $\quad Int(M) = M - \partial M$



A manifold with boundary and two charts.

Note that our previous notion of manifolds (manifolds <u>without</u> boundary) is included in this definition. Just as for them we define <u>charts</u>, <u>atlases</u>, <u>smooth atlases</u>, and <u>smooth manifolds with boundary</u>. Similarly, the notion of <u>smooth map</u> (check it on charts) and <u>diffeomorphism</u> generalize.

The definition we have given makes it hard to determine whether $x \in \partial M$, since we have to check all possible charts to see if x is in the image of $\partial \mathbb{R}^m_+$ for at least one of them. In fact, we need only check one chart, by the following Lemmas.

<u>Lemma</u>: If $U \subset \mathbb{R}^n_+$ is open, $\phi : U \to \mathbb{R}^m_+$ is smooth, $x_0 \in Int(U)$ and $\phi(x_0) \in \partial \mathbb{R}^m_+$, then

$$Im\,(D\phi(x_0)) \subset \partial \mathbb{R}^m_+ .$$

<u>Lemma</u>: If $U, V \subset \mathbb{R}^n_+$ are open and $f : U \to V$ is a diffeomorphism, f restricts to diffeomorphisms

$$\partial f : \partial U \to \partial V$$

and $Int\, f : Int\, U \to Int\, V$.

Proof of first lemma: Let $p : \mathbb{R}^m_+ \to \mathbb{R}$ be projection onto the last coordinate. Then $p\phi \geq 0$ on $\mathcal{U}$, and $p\phi(x_0) = 0$. For any t and v,

$$\phi(x_0 + tv) = \phi(x_0) + t\, D\phi(x_0)(v) + o(|tv|)$$

If $t \to 0^+$ we see that $pD\phi(x_0)(v) \geq 0$, and as $t \to 0^-$ we see the reverse, so $pD\phi(x_0)(v) = 0$. Thus $D\phi(x_0)(v) \in \ker(p) = \partial \mathbb{R}^m_+$. //

Proof of second lemma: The first lemma shows that if $Df(x_0)$ is an isomorphism and $x_0 \in \operatorname{Int} \mathcal{U}$ then $f(x_0) \in \operatorname{Int} V$. That is

$$f(\operatorname{Int} \mathcal{U}) \subset \operatorname{Int} V.$$

Similarly $f^{-1}(\operatorname{Int} V) \subset \operatorname{Int} \mathcal{U}$ and so $f(\operatorname{Int} \mathcal{U}) = \operatorname{Int}(V)$. //

Proposition: $\operatorname{Int}(M)$ is a manifold without boundary. ∂M is a submanifold of codimension 1, and is a manifold without boundary.

Pf: This is clear for $\operatorname{Int}(M)$. To see that ∂M is a submanifold of dimension $m-1$ observe that a chart $h : \mathcal{U} \to M$ with $x \in h(\partial \mathbb{R}^m_+)$ provides a chart $\partial h : \mathbb{R}^{m-1} \cong \partial \mathbb{R}^m_+ \to \partial M$, in a nhood of x in ∂M. These charts all have domain $\mathbb{R}^{m-1}$, so ∂M has no boundary. //

Definition: The tangent bundle $TM = \coprod_\alpha \mathcal{U}_\alpha \times \mathbb{R}^m / \sim$ just as for manifolds without boundary:

$$(x, v)_\alpha \sim (h_{\beta\alpha}(x), Dh_{\beta\alpha}(x)(v))_\beta \quad \text{if } x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

N.B. The tangent space $T_x M$ is $\mathbb{R}^m$ even if $x \in \partial M$. All possible tangents $v \in \mathbb{R}^m$ occur as tangents of curves which begin or end at x.

$T_x M \cong \mathbb{R}^m$

<u>Sard's Theorem</u>: If $M \xrightarrow{f} N$ is a smooth map from a smooth manifold with boundary M to a smooth manifold without boundary N, the set of points in N which are regular values for both f and ∂f is dense in N.

<u>Proof</u>: Critical values for f or ∂f are the union of two sets of measure zero, so has dense complement. //

<u>Transverse Intersection Theorem</u>: Suppose M is a smooth manifold with boundary, N is a smooth manifold without boundary, $f: M \to N$ is a smooth map, and $Z \subset N$ is a submanifold. If $f \pitchfork Z$ and $\partial f \pitchfork Z$ then $f^{-1}(Z)$ is a submanifold of M and

$$\partial(f^{-1}(Z)) = \partial M \cap f^{-1}(Z).$$

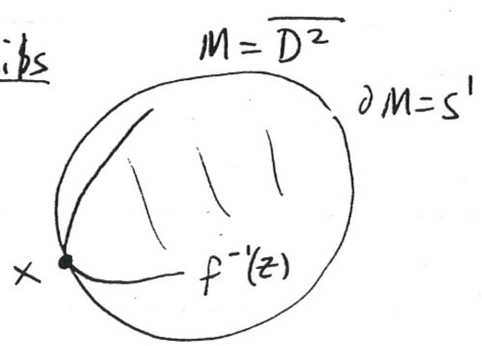
Codimension satisfies
$$\operatorname{codim}(f^{-1}(Z) \subset M) = \operatorname{codim}(\partial f^{-1}(Z) \subset \partial M)$$
$$= \operatorname{codim}(Z \subset N).$$

<u>Remark</u>: The condition $\partial(f^{-1}(Z)) = \partial M \cap f^{-1}(Z)$ can be split into

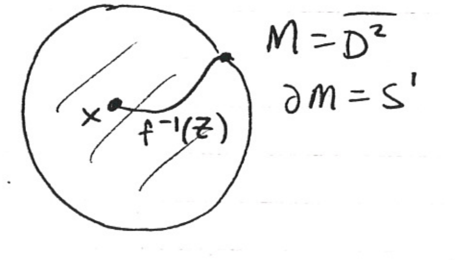
A: $\partial M \cap f^{-1}(Z) \subset \partial f^{-1}(Z)$
B: $\partial f^{-1}(Z) \subset \partial M \cap f^{-1}(Z)$, i.e. $\partial f^{-1}(Z) \subset \partial M$.

We can visualize what these are saying by thinking about what could happen if they failed.

<u>A fails</u>



<u>B fails</u>



Proof: We will show that $f \pitchfork Z$ implies B holds. Then we will prove a useful Lemma, and finally show that $\partial f \pitchfork Z$ implies condition A.

$f \pitchfork Z \Rightarrow B$: If $f \pitchfork Z$, $x \in \mathrm{Int}(M)$, and $f(x) \in Z$, then there exists a nhood of x such that

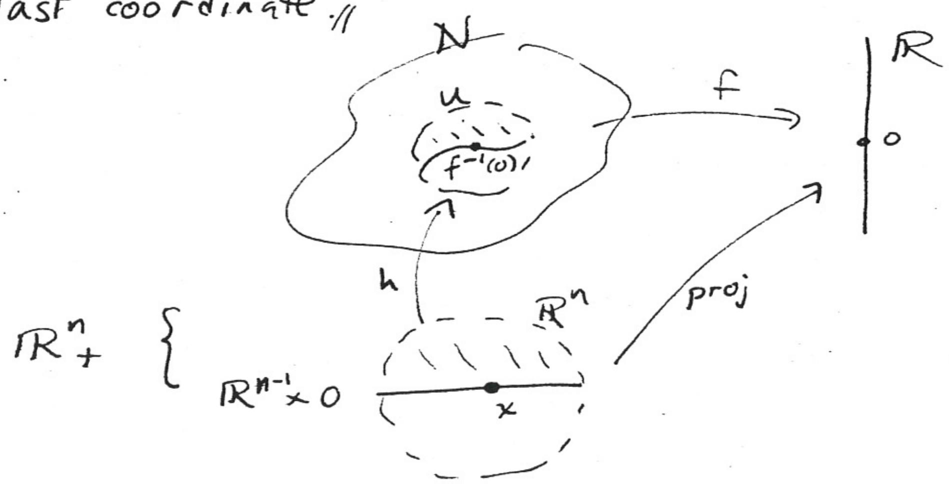
$$\begin{array}{ccccc} x & \in & f^{-1}(Z) \cap U & \cong & \mathbb{R}^{n-k} \times 0 \\ & & \cap & & \cap \\ & & U & \cong & \mathbb{R}^{n-k} \times \mathbb{R}^k \end{array}$$

This chart for $f^{-1}(Z)$ near x shows that $x \in \mathrm{Int}(f^{-1}(Z))$. That is, for $x \in f^{-1}(Z)$, if $x \notin \partial M$ then $x \notin \partial f^{-1}(Z)$, which is the contrapositive of B.//

The Lemma will also be useful in its own right. For example, it tells us that $\overline{D^n}$ is a manifold with $\partial \overline{D^n} = S^{n-1}$ by considering the function $f(x) = 1 - |x|^2$.

Lemma: Let N be a manifold without boundary and $f: N \to \mathbb{R}^1$ a smooth function. If 0 is a regular value for f then $f^{-1}(\mathbb{R}^1_+)$ is a manifold with boundary $f^{-1}(0)$.

Proof: The idea is the same as without boundary. Reduce to the local situation, $f: \mathbb{R}^n \to \mathbb{R}$ with 0 as a regular value. Then we have a chart about any $x \in f^{-1}(0)$ with respect to which f becomes projection onto the last coordinate.//



$\partial f \pitchfork Z \Rightarrow A$: Suppose $x \in \partial M \cap f^{-1}(Z)$. Choose charts about x and $f(x)$

$$\begin{array}{ccccc} h: \mathbb{R}^m_+ & \longrightarrow & U & \subset & M \\ k: \mathbb{R}^n & \longrightarrow & V & \subset & N \\ \cup & & \cup & & \cup \\ \mathbb{R}^{n-k} \times 0 & \longrightarrow & Z \cap V & \subset & Z \end{array}$$

such that $f(U) \subset V$. Now $Z \cap V = \pi^{-1}(0)$, where π is the projection

$$\begin{array}{ccc} \mathbb{R}^n & \xleftarrow{k^{-1}} & V \\ \pi_Z \downarrow & & \downarrow \pi \\ 0 \times \mathbb{R}^k & \cong & \mathbb{R}^k \end{array}.$$

Thus $f^{-1}(Z) = f^{-1}(\pi^{-1}(0)) = (\pi f)^{-1}(0)$. The conditions that f and ∂f be transverse to Z are equivalent to saying that πf and $\partial(\pi f)$ have 0 as a regular value. Replace f by πf.

The local model is therefore since we have already shown that $\partial f^{-1}(Z) \subset \partial M$.

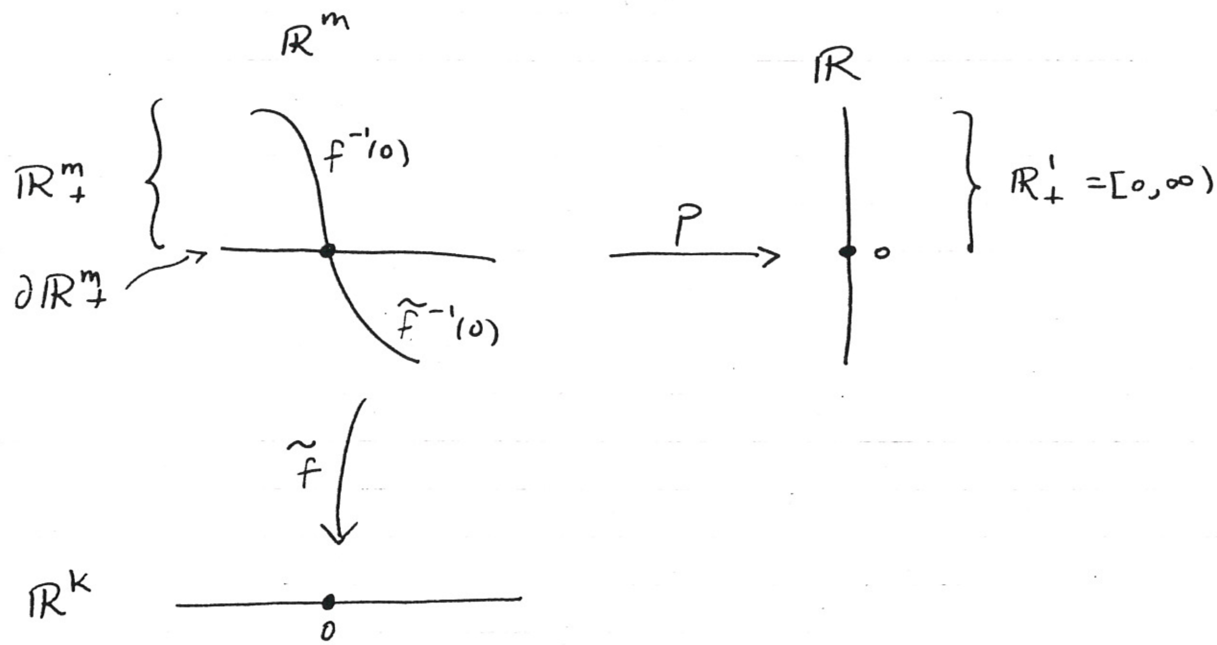
$$\begin{array}{ccccc} \partial \mathbb{R}^m_+ & \subset & \mathbb{R}^m_+ & \xrightarrow{f} & \mathbb{R}^k \\ \cup & & \cup & & \\ \partial f^{-1}(0) & \subset & f^{-1}(0) & & \end{array}$$

Now, smoothness of f means that f extends to a nhood of $\mathbb{R}^m_+$, which we may assume is $\mathbb{R}^m$. Say $\tilde{f}: \mathbb{R}^m \to \mathbb{R}^k$ with $f = \tilde{f} \mid \mathbb{R}^m_+$. By continuity, $\tilde{f} \pitchfork Z$ in some nhood of $\mathbb{R}^m_+$, so again passing to a smaller nhood if necessary, we may assume that $\tilde{f} \pitchfork Z$, i.e. $\tilde{f} \pitchfork 0$.

This implies that $\tilde{f}^{-1}(0)$ is a submanifold of $\mathbb{R}^m$ without boundary. Note that

$$f^{-1}(0) = \tilde{f}^{-1}(0) \cap \mathbb{R}^m_+ .$$

Let p be the restriction to $\tilde{f}^{-1}(0)$ of projection onto the last coordinate: $\mathbb{R}^m_+ \to \mathbb{R}$: $\qquad p : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$.



Thus $p : \tilde{f}^{-1}(0) \to \mathbb{R}$ satisfies $f^{-1}(0) = p^{-1}(\mathbb{R}^1_+)$ and

$$\partial M \cap f^{-1}(0) = p^{-1}(0),$$

so it is enough to show that 0 is a regular value for p, by the Lemma. On tangent spaces, we have, for $x \in \partial M \cap f^{-1}(0)$,

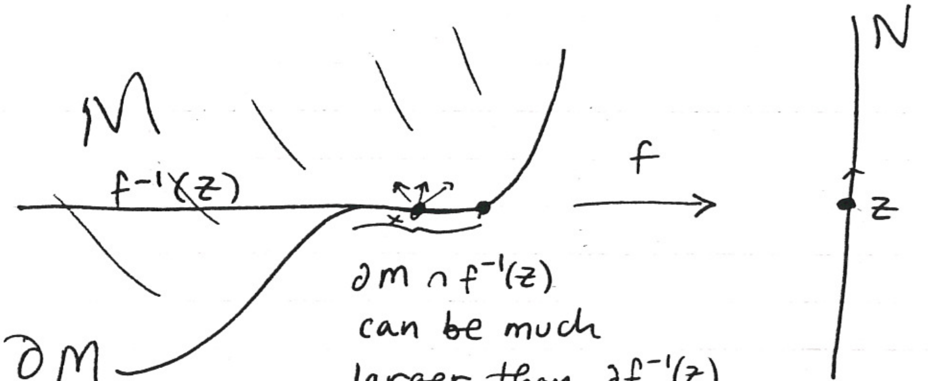
$$\begin{tikzcd}
 & T_x f^{-1}(0) = \ker T_x f \arrow[d] \arrow[dr, dotted, "?"] & \\
\ker T_x p = T_x \partial M \arrow[r, hook] \arrow[dr, "T_x \partial f"'] & T_x M \arrow[r, "T_x p"] \arrow[d, "T_x f"] & T_0 \mathbb{R} \\
 & T_0 N &
\end{tikzcd}$$

$T_x f$ is onto because $f \pitchfork Z$ and $T_x \partial f$ is onto since $\partial f \pitchfork Z$. We want to show $T_x p \,|\, T_x f^{-1}(0)$ is onto, i.e. nonzero. By linear algebra, $\dim(\ker T_x\partial f)$ is one less than $\dim(\ker T_xf)$ since $T_x\partial M$ has dimension one less than T_xM does. But this implies $\ker T_xf$ does not lie inside $T_x\partial M$, and this implies that $T_xp \mid T_xf^{-1}(0)$ is nonzero, and 0 is a regular value for

$$p : \tilde{f}^{-1}(0) \to \mathbb{R}.$$

Therefore $p^{-1}(0) = \partial M \cap \tilde{f}^{-1}(0) = \partial M \cap f^{-1}(0)$ is the boundary of $p^{-1}(\mathbb{R}^1_+) = f^{-1}(0)$, finishing the proof. //

The following example shows that the assumption that $\partial f \pitchfork Z$ is really necessary.



$\partial M \cap f^{-1}(z)$ can be much larger than $\partial f^{-1}(z)$ if $\partial f \not\pitchfork z$.

(f is transverse at x, but ∂f is not.)

By contrast, if $f \pitchfork Z$ and $\partial f \pitchfork Z$ we would have