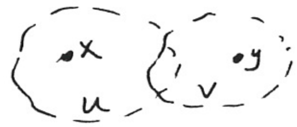
Review of point set topology

Let X be a topological space. Separation properties:

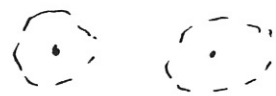
T_1 : Given distinct x and y in X $\exists$ open $\mathcal{U}$ and V such that $x \in \mathcal{U}$, $y \notin \mathcal{U}$ and $x \notin V$, $y \in V$



(each has a nhood not containing the other)

$T_1 \Leftrightarrow$ points are closed. (Zariski topology in alg. geom.)

T_2 (Hausdorff): Given distinct $x, y \in X$ $\exists$ disjoint open $\mathcal{U}$ and V s.t. $x \in \mathcal{U}$ and $y \in V$.



T_3 (regular) : T_1 + Given a closed set C and a point $x \notin C$ $\exists$ disjoint open sets containing x and C.

$T_{3\frac{1}{2}}$ (completely regular) : T_1 + Given a closed set C and a point $x \notin C$ $\exists$ a cont. function $f : X \to [0,1]$ s.t. $f(x) = 0$ and $f(C) = \{1\}$.

T_4 (normal) : T_1 and Given disjoint closed sets C and D there are disjoint open sets containing them.

Countability Properties

2nd countable : There is a countable basis for the topology.

Compactness properties

Def: Given a set C in a space X, an open cover of C is a collection $\{U_\alpha\}$ of open sets such that $C \subset \bigcup_\alpha U_\alpha$.

Def: C is compact if every open cover $\{U_\alpha\}$ has a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$, Lindelöf if $\exists$ a countable subcover.

Thm: (Munkres 4.2.4) Compact Hausdorff $\Rightarrow$ normal.

Thm: (Munkres 4.2.5) Regular with a countable basis $\Rightarrow$ normal.

Def: A collection $\mathcal{V} = \{V_\beta\}$ refines a collection $\{U_\alpha\}$ if each V_β is contained in some U_α.

A collection $\{U_\alpha\}$ is locally finite if for each $x \in X$ there is a nhood N_x which intersects only finitely many U_α.

X is paracompact if every open covering has a locally finite refinement which is an open covering.

Thm: If X is connected, locally Euclidean, and Hausdorff, then TFAE

(1) 2^{nd} countable
(2) metrizable
(3) paracompact
(4) σ-compact (i.e. countable union of compact)
(5) Lindelöf

We will see more of paracompactness later. (See Munkres Ch. 6)

Thm (Munkres 6.4.1) Paracompact and Hausdorff $\Rightarrow$ normal.

Some important theorems

Urysohn Lemma: If A and B are disjoint closed sets in a normal space X, there exists a continuous function $f: X \longrightarrow [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Urysohn metrization theorem: Regular and 2^{nd} countable $\Rightarrow$ metrizable.

Tietze extension theorem: Let A be a closed subset of a normal space X. Then

(a) any continuous map $A \to [a,b]$ can be extended to a map $X \to [a,b]$, and

(b) any continuous map $A \to \mathbb{R}$ can be extended to a map $X \to \mathbb{R}$.

Some useful facts.

1. If gf is 1-1 then f is 1-1.

2. If gf is onto then g is onto.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

3. $f: X \to Y$ is 1-1 and onto

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$$

$\Longleftrightarrow$ $\exists\, g: Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

4. A continuous function $f: X \to Y$ is a homeomorphism $\Longleftrightarrow$ $\exists$ a continuous map $g: Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

Notation: 1_X is the identity map of X.

Def: The disjoint union $\coprod_{i \in I} X_i$ of a collection $\{X_i\}_{i \in I}$ of topological spaces is the subspace $\bigcup_{i \in I} (X_i \times \{i\})$ of the product $(\bigcup_i X_i) \times I$, where I is given the discrete topology. There is a natural inclusion $\imath_i : X_i \longrightarrow \coprod_{i \in I} X_i$ given by $\imath_i(x) = (x, i)$ for each $i \in I$.

<u>Prop</u>: (1) Given functions $f_i : X_i \to Y$ for each $i \in I$ there exists a unique function
$$f : \coprod_{i \in I} X_i \longrightarrow Y$$
such that $f \iota_i = f_i$ for each $i \in I$.

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \coprod_{i \in I} X_i \arrow[d, dashed, "f"] \\ & Y \end{tikzcd}$$

(2) f is continuous iff f_i is continuous for each $i \in I$.

There is a natural quotient map $\coprod_i X_i \xrightarrow{p} \bigcup_i X_i$ defined by letting $p\iota_i : X_i \to \bigcup_i X_i$ be the inclusion.

<u>Prop</u>: $\coprod_i X_i$ is an m-manifold iff each X_i is an m-manifold.

<u>Remark</u>: The Cartesian product $\prod_i X_i$ and its projection maps $p_i : \prod_i X_i \to X_i$ satisfy properties dual to the $\iota_i : X_i \to \coprod_i X_i$.

(1) Given functions $\{ Y \xrightarrow{f_i} X_i \mid i \in I \}$ there exists a unique function $Y \xrightarrow{f} \Pi X_i$ such that $f_i = p_i f \; \forall i$.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_i X_i \arrow[d, "p_i"] \\ & X_i \end{tikzcd}$$

(2) f is continuous $\iff$ each f_i is continuous.

This is the reason the product topology is the "correct" topology to put on the Cartesian product.