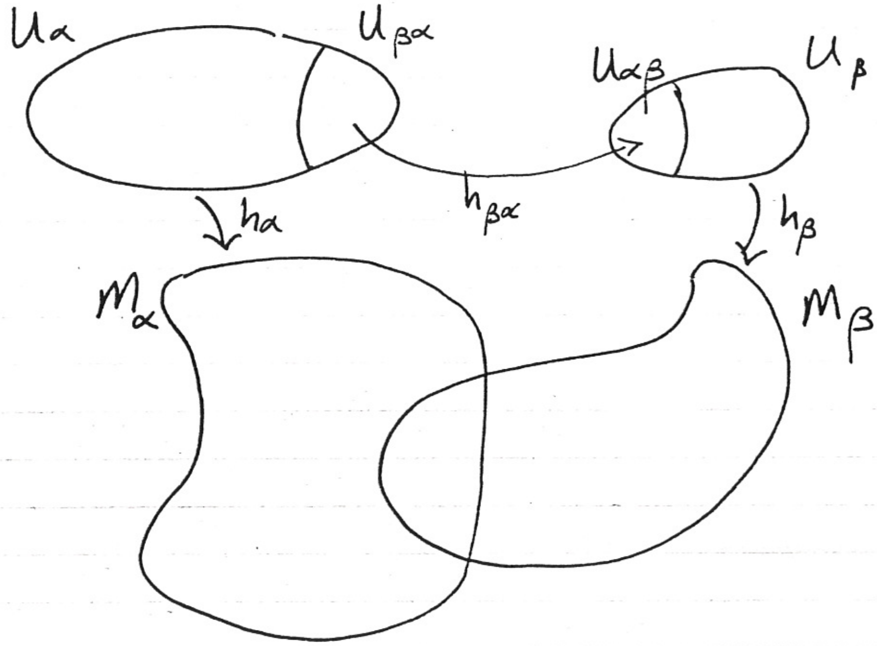
Reconstructing a manifold M from an atlas

Let $\mathcal{H} = \{h_\alpha\}_{\alpha \in A}$ be an atlas for M. This means we have

homeomorphisms $h_\alpha : U_\alpha \longrightarrow M_\alpha$,

U_α open in $\mathbb{R}^m$, M_α open in M,

$$M = \bigcup_{\alpha \in A} M_\alpha .$$



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Let $h_{\beta\alpha} = h_\beta^{-1} h_\alpha : U_{\beta\alpha} \longrightarrow U_{\alpha\beta}$. Note that $h_{\beta\alpha}$ is a homeomorphism with inverse $h_{\alpha\beta}$. The open sets $\{U_\alpha\}$ and the homeomorphisms $\{h_{\alpha\beta}\}$ completely determine the manifold M as follows.

Given the open subsets $\{U_\alpha\}_{\alpha \in A}$ of $\mathbb{R}^n$ and the homeomorphisms $\{h_{\alpha\beta}\}$, let

$$X = \coprod_{\alpha \in A} U_\alpha$$

and let $i_\alpha : U_\alpha \longrightarrow X$ be the natural inclusion, so that a function $X \xrightarrow{f} Y$ is continuous iff each $f i_\alpha$ is continuous. Let

$$h : X \longrightarrow M$$

be the map defined by $i_\alpha = h_\alpha : U_\alpha \longrightarrow M$. Define an equivalence relation $\sim$ on X by

$$a \sim b \quad \text{iff} \quad h(a) = h(b)$$

that is,

$$\begin{aligned} i_\alpha(x) \sim i_\beta(x') \quad &\Leftrightarrow \quad h i_\alpha(x) = h i_\beta(x') \\ &\Leftrightarrow \quad h_\alpha(x) = h_\beta(x') \\ &\Leftrightarrow \quad x' = h_{\beta\alpha}(x) \,. \end{aligned}$$

From the latter version of the equivalence relation, we see that it is determined solely by the $\{h_{\alpha\beta}\}$, without reference to M or h. Let $M' = X/\sim$ with quotient map $q : X \longrightarrow M'$. We have the setup

$$\begin{tikzcd} U_\alpha \arrow[r, "i_\alpha"'] \arrow[rr, bend left, "h'_\alpha"] \arrow[dr, bend right, "h_\alpha"'] & X \arrow[r, "q"] \arrow[d, "h"] & M' \arrow[dl, dotted, "h'"] \\ & M & \end{tikzcd}$$

where $h'_\alpha = q i_\alpha$.

<u>Claim</u>: M' is a manifold with charts $\{h'_\alpha\}$.

Further, since $h(a) = h(b)$ if $a \sim b$, h induces a continuous map $h' : M' \to M$, by $h' q\, i_\alpha(x) = h_\alpha(x)$.

<u>Prop</u>: h' is a homeomorphism.

<u>Pf</u>: h' is well defined since $q i_\alpha(x) = q i_\beta(x') \Leftrightarrow i_\alpha(x) \sim i_\beta(x')$
$\Leftrightarrow h_\alpha(x) = h_\beta(x')$.

h' is continuous since q is a quotient map and $h'q = h$ is continuous.

h' is 1-1 since $h'q(a) = h'q(b) \Rightarrow h(a) = h(b) \Rightarrow a \sim b \Rightarrow qa = qb$.

h' is onto since $M = \bigcup_\alpha \operatorname{im}(h_\alpha)$ and $h_\alpha(x) = h' q\, i_\alpha(x)$.

h is open because any open set in X is the disjoint union, $\coprod_\alpha i_\alpha(V_\alpha)$, with $V_\alpha \subset U_\alpha$ open. Then $h i_\alpha(V_\alpha) = h_\alpha(V_\alpha)$ is open since h_α is a homeomorphism to its image, and that image is open in M.

h' is open because $U \subset M'$ is open iff $q^{-1}U$ is open in X, so $h'(U) = h' q q^{-1}(U) = h q^{-1}(U)$, which is open since h' is open. //