Regular Values and Transversality

Def: Let $f: M \to N$ be a smooth map.

(i) $y \in N$ is a regular value of f if $T_x f$ is onto for all $x \in f^{-1}(y)$.

(ii) $x \in M$ is a critical point and $f(x) \in N$ is a critical value if $T_x f$ is not onto.

Notes: (1) Each $y \in N$ is either a critical value or regular value according as $f^{-1}(y)$ does or does not contain a critical point.

(2) If y is not in the image of f then y is a regular value.

Theorem: If $y \in N$ is a regular value then $f^{-1}(y)$ is a submanifold of M with $T_x f^{-1}(y) = \ker(T_x f)$.

Proof: Second countability and the Hausdorff condition are inherited from M, so we need only produce charts for $f^{-1}(y)$. Suppose $x \in f^{-1}(y)$. By the Inverse Function Theorem there are charts $h: U \to U' \subseteq M$ and $k: V \to V' \subseteq N$, $x \in U'$, $y \in V'$, such that $k^{-1} f h$ is the

$$\begin{tikzcd}
x \in f^{-1}(y) \arrow[r, phantom, "\subset"] & U' \arrow[r, "f"] & V' \arrow[d, "k^{-1}"] \\
U \cap (0 \times \mathbb{R}^{m-n}) \arrow[u] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap" description] & U \arrow[u, "h"] \arrow[r, "k^{-1} f h"] \arrow[d, phantom, "\cap" description] & V \arrow[d, phantom, "\cap" description] \\
0 \times \mathbb{R}^{m-n} \arrow[r, phantom, "\subset"] & \mathbb{R}^n \times \mathbb{R}^{m-n} \arrow[r, "(I\ 0)"'] & \mathbb{R}^n
\end{tikzcd}$$

projection $(I\ 0): \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$. Then $h | U \cap (0 \times \mathbb{R}^{m-n})$ is a chart for $f^{-1}(y)$ in a nhood of x. The tangent space to $U \cap (0 \times \mathbb{R}^{m-n})$ is $0 \times \mathbb{R}^{m-n} = \ker(I\ 0)$ and Th maps this to $\ker(T_x f)$. //

The theorem allows a simple proof that S^n is a manifold: $1 \in \mathbb{R}$ is a regular value of the function $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ which sends x to $x \cdot x = |x|^2$. It also implies that $T_x S^n$ is the kernel of $v \mapsto 2x \cdot v$, the subspace perpendicular to x.

Note that

$$\dim(f^{-1}(y)) + \dim N = \dim M.$$

Corollary: The following are Lie groups and their Lie algebras:

G	$T_e G$	
$Gl_n(k)$, $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$	$M_n(k)$	
$Sl_n(k)$, $k = \mathbb{R}$ or $\mathbb{C}$	$\{A \in M_n(k) \mid \operatorname{tr} A = 0\}$	(trace zero)
$O(n)$	$\{A \in M_n(\mathbb{R}) \mid A + A^t = 0\}$	(skew-symmetric)
$U(n)$	$\{A \in M_n(\mathbb{C}) \mid A + \bar{A}^t = 0\}$	(skew-Hermitian)
$Sp(n)$	$\{A \in M_n(\mathbb{H}) \mid A + \bar{A}^t = 0\}$	
$SO(n)$	$\{A \in M_n(\mathbb{R}) \mid A + A^t = 0\}$, same as $O(n)$	
$SU(n)$	$\{A \in M_n(\mathbb{C}) \mid A + \bar{A}^t = 0$ and $\operatorname{tr} A = 0\}$	

Proof: $Gl_n(k)$ is an open subset of $M_n(k) = \mathbb{R}^{dn^2}$, $d = 1, 2$ or 4. Each of the others is the inverse image of a regular value, so it is sufficient these functions, their derivatives, and a regular value.

$Sl_n(k) = \det^{-1}(1)$, $\det : M_n(k) \longrightarrow k$, $D\det(I) = \operatorname{tr}$.

$O(n) = f^{-1}(I)$, $f : Gl_n(\mathbb{R}) \longrightarrow Sym_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A = A^t\}$ by $f(A) = AA^t$. $Df(I)(A) = A + A^t$.

$U(n) = f^{-1}(I)$, $f : Gl_n(\mathbb{C}) \rightarrow Sym_n(\mathbb{C})$ by $f(A) = A\bar{A}^t$. $Df(I)(A) = A + \bar{A}^t$.

$Sp(n)$ as for $U(n)$ with $\mathbb{C}$ replaced by $\mathbb{H}$.

$SO(n)$ is the component of I in $O(n)$, since $AA^t = I$ implies that $\det(A)^2 = 1$, so $O(n)$ is the disjoint union of $SO(n)$, with determinant 1, and the coset $A \cdot SO(n)$ with determinant -1, where A is any orthogonal matrix with determinant -1. Since $SO(n)$ is open in $O(n)$, they have the same tangent space.

$SU(n)$ In the complex case, the condition $\det(A) = 1$ does reduce the dimension of $U(n)$ by 1, since

$$\det : U(n) \longrightarrow S^1$$

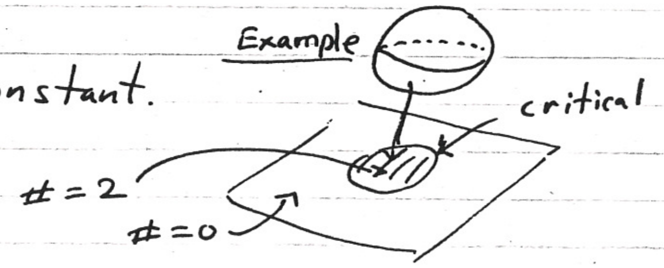
has $1 \in S^1$ as a regular value. //

Following Milnor [Topology from the Differentiable Viewpoint] we can give a very nice proof of the fundamental theorem of algebra. First, we need a general observation.

Suppose $M \xrightarrow{f} N$ is a smooth map between manifolds of the same dimension, and that M is compact. Then, if $y \in N$ is a regular value, $f^{-1}(y)$ is a 0 dimensional submanifold of M, and therefore discrete, and is also compact, being closed in M. Therefore

$$\# f^{-1}(y) = \text{the number of points in } f^{-1}(y)$$

is finite. Now, if $x \in f^{-1}(y)$, y regular, then $T_x f$ is onto, and therefore an isomorphism since $\dim M = \dim N$. Hence f is a diffeomorphism from a nhood of x to a nhood of y. This proves



Lemma: $\# f^{-1}(y)$ is locally constant.

The Fundamental Theorem of Algebra:

Every nonconstant complex polynomial has a root.

Proof: Suppose $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree > 0. Then $|p(z)| \to \infty$ as $|z| \to \infty$, so we may extend p to a continuous map

$$\begin{array}{ccc} S^2 & \xrightarrow{\bar{p}} & S^2 \\ \| & & \| \\ \mathbb{C} \cup \infty & & \mathbb{C} \cup \infty \end{array}$$

by $\bar{p}(\infty) = \infty$. That this is smooth at ∞ follows by examining

$$\frac{1}{\bar{p}\left(\frac{1}{z}\right)}$$

in a nhood of 0. Now, critical points of p are zeros of the derivative, so there are finitely many. Therefore all but a finite number of points in S^2 are regular values. Thus, the regular values form a connected open set. Since $\# f^{-1}(y)$ is locally constant on a connected set, it is constant.

Now, $\# f^{-1}(y)$ cannot be zero almost everywhere, therefore it is nonzero almost everywhere. Therefore, $\bar{p}$ is onto. (At critical values, consider a sequence tending toward them.) Since $\bar{p}$ is onto, it has a root. //

When we consider Sard's Theorem, we will see that the behavior of the polynomial is not unusual: the set of regular values is always dense.

First, we extend the notion of a regular value.

<u>Def</u>: A map $M \xrightarrow{f} N$ is <u>transverse</u> to a submanifold $Y \subset N$ if for each $x \in f^{-1}(Y)$ we have

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & & \cup \\ & & Y \end{array}$$

$$T_x f(T_x M) + T_{f(x)} Y = T_{f(x)} N.$$

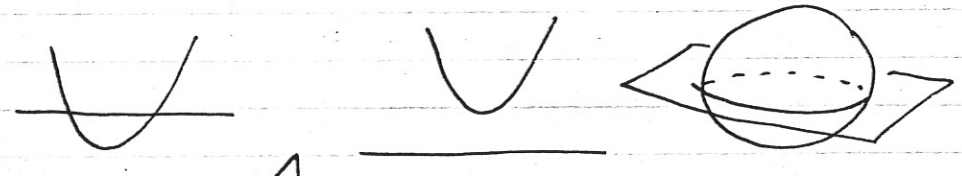
That is, the images of tangents to M together with tangents to Y span the entire tangent space of N.

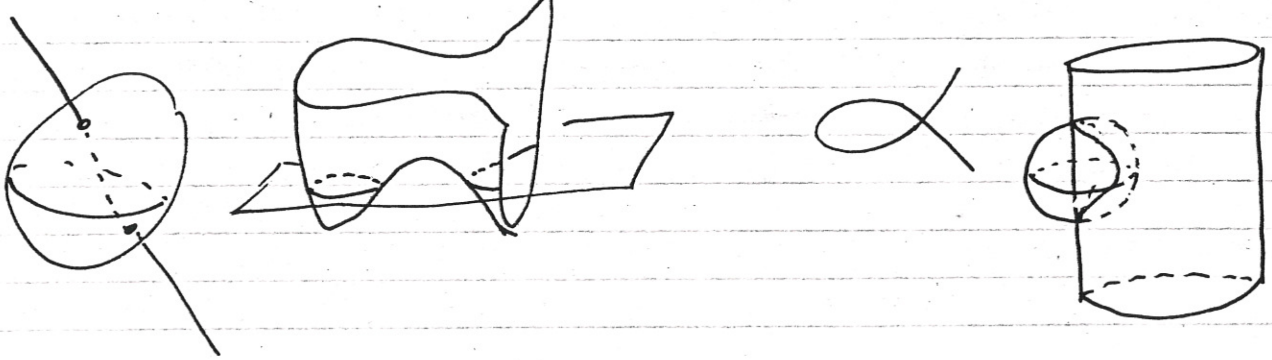
Write $f \pitchfork Y$ to denote this.

<u>Def</u>: Two submanifolds $Y_1, Y_2 \subset N$ are <u>transverse</u>, written $Y_1 \pitchfork Y_2$, if the inclusion $i_1 : Y_1 \to N$ is transverse to Y_2. Equivalently,

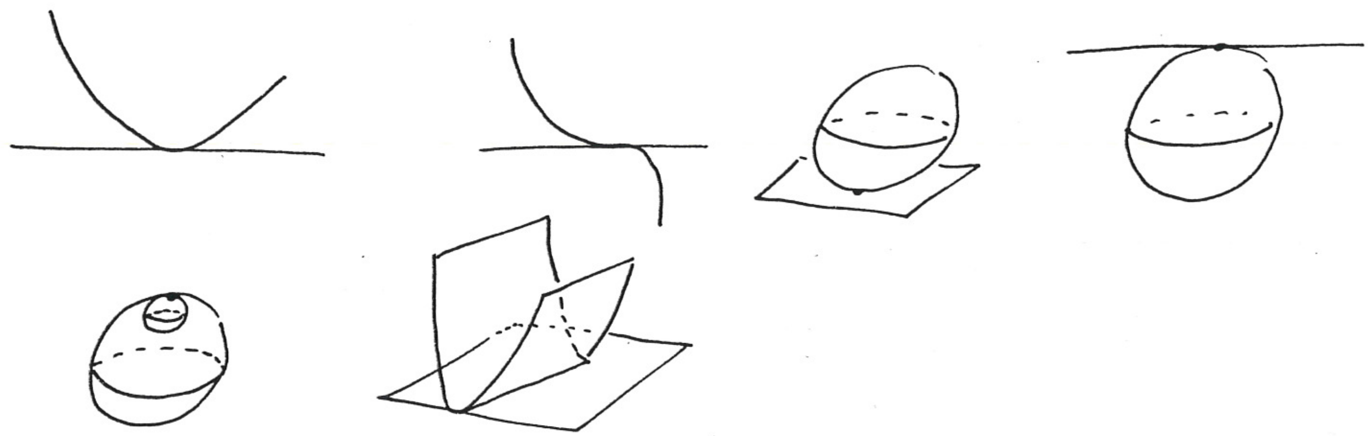
$$T_y Y_1 + T_y Y_2 = T_y N \qquad \text{for each } y \in Y_1 \cap Y_2$$

<u>Examples</u>:





Transverse Intersections



Not transverse

<u>Note</u>: $f \pitchfork \{y\} \iff y$ is a regular value of f.

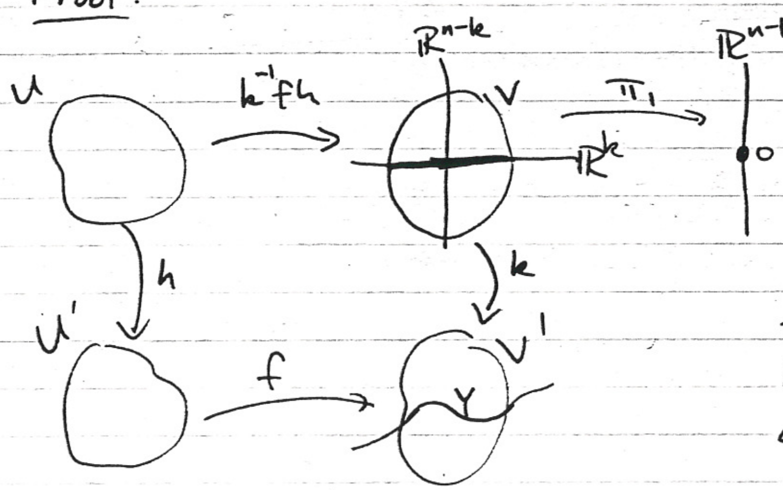
<u>Theorem</u>: If $f \pitchfork Y$ then $f^{-1}(Y)$ is a submanifold of M,

$$\operatorname{codim}(f^{-1}(Y)) = \operatorname{codim}(Y), \text{ and}$$

$$T_x f^{-1}(Y) = \ker\left(T_x M \xrightarrow[T_x f]{} T_{f(x)} N \longrightarrow T_{f(x)} N \big/ T_{f(x)} Y\right)$$

$$= (T_x f)^{-1}(T_{f(x)} Y)$$

<u>Proof</u>:



Choose charts $h: U \to U'$, $x \in U'$, and $k: V \to V'$, $y = f(x) \in V'$, such that $f(U') \subset V'$, and

$$k^{-1}(Y \cap V') = (\mathbb{R}^k \times 0) \cap V$$

Then 0 is a regular value for $\pi_1 k^{-1} f h$ and $(k^{-1}fh)^{-1}(\mathbb{R}^k \times 0) = h^{-1}(f^{-1}(Y))$ is the same as $(\pi_1 k^{-1} f h)^{-1}(0)$, so is a submanifold. //

Locally, we have written Y as the zero set of $n-k$ independent functions (the coordinate functions of $\mathbb{R}^{n-k}$ composed with $\pi_1 k^{-1}$), say $g_1, \dots, g_{n-k}$. Then $f^{-1}(Y)$ is expressed as the zero set of the $n-k$ independent functions $g_1 \circ f, \dots, g_{n-k} \circ f$. This shows $\operatorname{codim}(f^{-1}(Y)) = \operatorname{codim}(Y)$.

<u>Cor</u>: If $Y_1 \pitchfork Y_2$ then $Y_1 \cap Y_2$ is a submanifold of N and

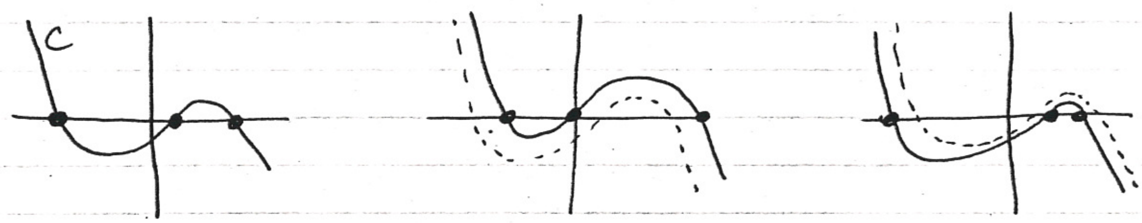
$$\operatorname{codim}(Y_1 \cap Y_2) = \operatorname{codim}(Y_1) + \operatorname{codim}(Y_2).$$

<u>Note</u>: Transversality depends upon the ambient space N. For example $\mathbb{R} \times 0$ and $0 \times \mathbb{R}$ intersect transversally in $\mathbb{R}^2$, but not in $\mathbb{R}^3$. In general, if

$$\dim(Y_1) + \dim(Y_2) < \dim(N)$$

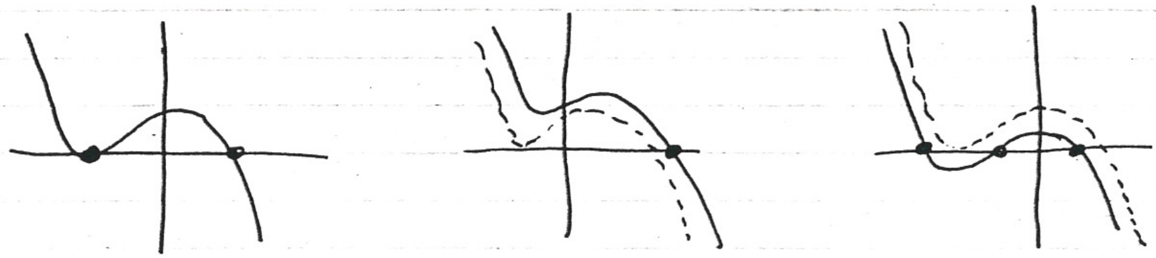
then Y_1 and Y_2 can only intersect transversally if they do not intersect at all.

The importance of transverse intersections is that they are <u>stable</u>, that is, invariant under small perturbations. For example:



$C \pitchfork \mathbb{R} \times 0$

Transverse intersection



$C \not\pitchfork \mathbb{R} \times 0$ Non-transverse intersection