Riemannian Metrics

We will show that every manifold has a Riemannian metric and deduce some useful consequences.

Def: An inner product $\langle\ ,\ \rangle$ on a vector space V is a symmetric bilinear function

$$\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}.$$

(Equivalently, $\langle -,- \rangle$ is a symmetric linear map $V \otimes V \longrightarrow \mathbb{R}$.)

$\langle -,- \rangle$ is positive definite if $\langle v,v \rangle \geq 0$ for all $v \in V$ and

$$\langle v,v \rangle = 0 \iff v = 0.$$

Note: If $\{v_i\}$ is a basis for V and $A_{ij} = \langle v_i, v_j \rangle$ then, writing vectors $v, w \in V$ as column vectors with respect to the basis $\{v_i\}$ the inner product is given by the matrix $A = (A_{ij})$ by

$$\langle v, w \rangle = v^t A w.$$

Def: A Riemannian metric on a manifold M is a positive definite inner product $\langle -,- \rangle_m$ on $T_m M$ for each $m \in M$, varying smoothly with m.

Note: There are three ways in which we could express smoothness.

1. if X and Y are smooth tangent vector fields, then $\langle X,Y \rangle$ is a smooth function on M. ($\langle X,Y \rangle(m) = \langle X(m), Y(m) \rangle_m$ here.)

2. if $U_i \xrightarrow{h} M$ is a chart with induced chart $U_i \times \mathbb{R}^m \xrightarrow{Th} TM$, and if we take as our basis of $T_m M$ the vectors $Th(u, e_j)$ where e_j, $j = 1, \dots, m$, is the j^{th} standard basis vector of $\mathbb{R}^m$, then the associated matrix $A(m)$ defines a smooth function

$$U_i \xrightarrow{A} M_n(\mathbb{R})$$

3. Let $\Delta_2 \subset TM \times TM$ be the subset $\Delta_2 = \{(v_1, v_2) \mid \tau v_1 = \tau v_2\}$, that is v_1 and v_2 lie is the same tangent space. This is a smooth submanifold of $TM \times TM$ because it is $(\tau \times \tau)^{-1}(\Delta)$, where $\Delta \subset M \times M$ is the diagonal, $\Delta = \{(m,m) \mid m \in M\}$, and, since τ is a submersion, $\tau \times \tau$ is transverse to Δ.

The space Δ_2 is the natural domain of $\langle -,- \rangle$,

$$\Delta_2 = \bigcup_{x \in M} T_x M \times T_x M \xrightarrow{\langle -,- \rangle} \mathbb{R}$$

and we require that $\langle -,- \rangle : \Delta_2 \to \mathbb{R}$ be smooth.

Prop: Condition (1) - (3) for smoothness are equivalent.
Proof: Easy. Exercise. //

Given a Riemannian metric, we can measure the lengths of paths:

Def: If $\alpha : [a,b] \to M$ is smooth, let the <u>length of α</u> be

$$\ell(\alpha) = \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle_{\alpha(t)}}\, dt = \int_a^b \sqrt{\langle \alpha', \alpha' \rangle}\, dt$$

Clearly $\ell(\alpha)$ is additive on intervals: if $c \in [a,b]$ and $\alpha_1 = \alpha | [a,c]$ and $\alpha_2 = \alpha | [c,b]$ then $\ell(\alpha) = \ell(\alpha_1) + \ell(\alpha_2)$. Thus, we may also define the length of piecewise smooth curves.

The <u>induced metric</u> on M is

$$d(m,n) = \inf \{ \ell(\alpha) \mid \alpha \text{ is a piecewise smooth curve from } m \text{ to } n \}.$$

Prop: d is a metric on M.

Proof: Clearly d satisfies the triangle inequality, is symmetric, and has $d(m,m) = 0$ and $d(m,n) \geq 0$. To show $d(m,n) > 0$ if $m \neq n$, consider a chart around m and any path α from m to n.

If we can show that there is a lower bound for the length of the part of the path within the chart, we are done. So, we may assume α lies entirely within a chart. Since the image of α is compact, there are constants $c_1, c_2 > 0$ such that

$$(*) \qquad c_1 \|\alpha'(t)\| \leq \sqrt{\langle \alpha'(t), \alpha'(t) \rangle_{\alpha(t)}} \leq c_2 \|\alpha'(t)\|$$

where $\|\cdot\|$ is the usual Euclidean norm. Then

$$\begin{aligned} \ell(\alpha) &= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle_{\alpha(t)}}\, dt \\ &\geq c_1 \int_a^b \|\alpha'(t)\|\, dt \\ &\geq c_1 \left\| \int_a^b \alpha'(t)\, dt \right\| \\ &= c_1 \|\alpha(b) - \alpha(a)\| > 0 \end{aligned}$$

Choosing c_1 so that $(*)$ holds in a compact subset of the chart with $\alpha(a)$ and $\alpha(b)$ in it, we obtain a lower bound for $\ell(\alpha)$ which depends only on $\alpha(a)$ and $\alpha(b)$. //

<u>Lemma</u>: If $\langle -,- \rangle_i$ is a positive definite inner product for $i = 1, \dots, m$, and if $c_i \geq 0$ for $i = 1, \dots, m$ with $\sum c_i > 0$ then

$$\langle v, w \rangle = \sum_i c_i \langle v, w \rangle_i$$

is a positive definite inner product.

<u>Proof</u>: Easy. //

<u>Theorem</u>: Every manifold has a Riemannian metric.

<u>Proof</u>: The chart $U_i \times \mathbb{R}^m$ has a Riemannian metric $\langle -,- \rangle_i$ given by the usual inner product in $\mathbb{R}^m$. Let $\{\phi_i\}$ be a partition of unity subordinate to an atlas and let

$$\langle -,- \rangle = \sum_i \phi_i \langle -,- \rangle_i \, . \quad //$$

Riemannian metrics are <u>contravariant</u>: they "pull back" along maps rather than "pushing forward" as do tangent vectors, which are thereby <u>covariant</u>. More precisely:

<u>Prop</u>: If M is a submanifold of N and N has a Riemannian metric, then M inherits a Riemannian metric with respect to which the inclusions $T_m M \to T_m N$ are isometries.

<u>Proof</u>: Note that we really only need an immersion $M \hookrightarrow N$ since it is the tangent map $T_m M \to T_m N$ which must be one-to-one to make the restricted inner product positive definite. The restricted inner product $\langle -,- \rangle_m \mid T_m M \times T_m M \to T_m N \times T_m N \to \mathbb{R}$ is clearly a Riemannian metric.//

<u>Def</u>: If M is a Riemannian manifold then the <u>unit sphere</u> and <u>unit disk tangent bundles</u> are, respectively,

$$S(m) = \{ v \in T(m) \mid \|v\| = 1 \}$$

$$\cap\!|$$

$$D(m) = \{ v \in T(m) \mid \|v\| \leq 1 \}$$

where $\|v\| = \sqrt{\langle v, v \rangle_{\tau(v)}}$.

<u>Note</u>: Similarly we can define sphere and disk bundles

$$S_\varepsilon(M) \subset D_\varepsilon(M) \quad \text{for any positive function } \varepsilon : M \to \mathbb{R},$$

and $S_\varepsilon(m)$ is diffeomorphic to $S(m)$ for all ε. Similarly, $D_\varepsilon(m)$ will be seen to be diffeomorphic to $D(m)$ when we study manifolds with boundary. This comment presumes the following.

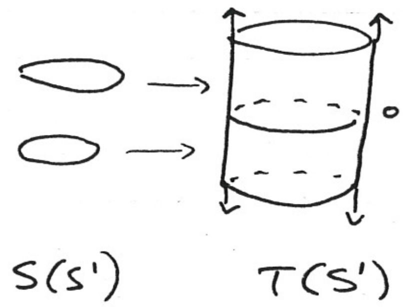
<u>Theorem</u>: $S(m)$ is a smooth $2m-1$ submanifold of TM.

<u>Proof</u>: 1 is a regular value for the function $v \mapsto \|v\|^2$.//

<u>Example 1:</u> $T(S^1) \cong S^1 \times \mathbb{R}$

$S(S^1) \cong S^1 \times \{\pm 1\}$

$\cong SO(2)$

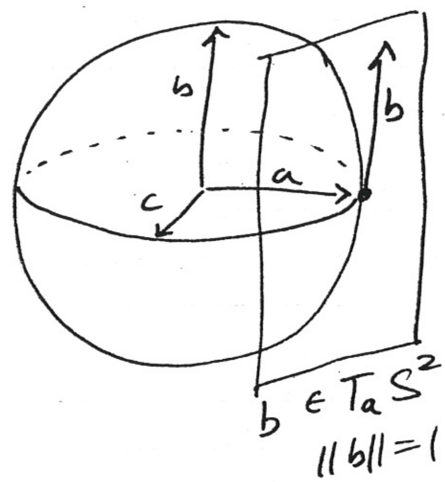


<u>Example 2:</u> $S(S^2) \cong SO(3) \cong \mathbb{RP}^3$

If $A = (a \ b \ c) \in SO(3)$

then $a \in S^2$, and $T_a S^2$ is the plane perpendicular to a, so b is a unit vector in $T_a S^2$.

This defines a diffeomorphism $SO(3) \longrightarrow S(S^2)$ since $c = a \times b$ is completely determined by a and b.



<u>Example 3:</u> In general, $S(S^{n-1})$ is the <u>Stiefel manifold</u>

$$V_{n,n-2} = SO(n)/SO(n-2),$$

the coset space of the natural inclusion of $SO(n-2)$ into $SO(n)$. A point of $V_{n,n-2}$ is a pair (v, w) of unit vectors in $\mathbb{R}^n$ which are orthogonal. These form the first two columns of an element of $SO(n)$, or a point of S^{n-1} together with a unit tangent vector at that point.

Another useful construction using Riemannian metrics is the <u>normal bundle</u> of an embedding, i.e. submanifold.

<u>Def:</u> If N is a submanifold of a Riemannian manifold M, <u>the normal bundle of N in M</u> is

$$\nu(N \subset M) = \nu_{N \subset M} = \{ v \in T_x M \mid v \perp T_x N, \ x \in N \}.$$

Thus, at each point of the submanifold N we have an orthogonal decomposition of the tangent space of the larger

$$T_x M \cong T_x N \oplus \nu(N \subset M)$$

manifold into vectors tangent to and normal to the submanifold, respectively. Briefly, $TM|_N = TN \oplus \nu$.

Example 1: $\nu(S^n \subset \mathbb{R}^{n+1}) = \nu(S^n \subset S^{n+1}) \cong S^n \times \mathbb{R}$

The first equality follows from the fact that the normal bundle depends only on an open neighborhood of the submanifold. The second follows from the fact that a normal vector at $x \in S^n$ is a multiple rx for some $r \in \mathbb{R}$.

Since the tangent bundle of $\mathbb{R}^{n+1}$ is trivial, we have the curious situation that TS^n may not be trivial, yet after adding a trivial bundle $(\nu(S^n \subset \mathbb{R}^{n+1}))$ the result is trivial. We call such bundles stably trivial.

Example 2: The canonical line bundle λ over $\mathbb{R}P^n$ is the space

$$E = \{ (\ell, x) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid x \in \ell \}$$

where we think of a point in $\mathbb{R}P^n$ as a line in $\mathbb{R}^{n+1}$. There is an obvious projection map $\lambda: E \longrightarrow \mathbb{R}P^n$, $\lambda(\ell, x) = \ell$, and λ^{-1} of any point is a one dimensional vector space.

Prop: $\lambda = \nu(\mathbb{R}P^n \subset \mathbb{R}P^{n+1})$

Proof: Exercise. //

There is an alternative construction of $\nu(N \subset M)$ which assigns, at $x \in N$, the vector space

$$T_x M / T_x N.$$

Clearly
$$\nu_x \subset T_x M \longrightarrow T_x M / T_x N$$
is an isomorphism. The advantage of this construction of ν is that it is clearly independent of any Riemannian metric; its disadvantage is that we would need to discuss vector bundles first.

<u>Theorem</u>: $\nu(N \subset M)$ is a dim(M)-manifold containing N.

<u>Tubular Neighborhood Theorem</u>: There is a diffeomorphism
$$\nu(N \subset M) \xrightarrow{f} \mathcal{U}$$
to an open nhood $\mathcal{U}$ of N in M, such that the inclusion of N in M factors as the 0-section of ν followed by f.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "0"'] & M \arrow[d, phantom, "\cup" description] \\ \nu(N \subset M) \arrow[r, "f", "\cong"'] & \mathcal{U} \end{tikzcd}$$

<u>Proofs</u>:

The condition that N be a submanifold of M is that we can find charts $\mathcal{U}_i \to M$ which restrict to charts of N

$$\begin{array}{ccc} \mathcal{U}_i & \longrightarrow & M \\ \cup & & \cup \\ \mathcal{U}_i \cap (\mathbb{R}^n \times 0) & \longrightarrow & N \end{array}$$

Consider the induced chart $\mathcal{U}_i \times \mathbb{R}^m \longrightarrow TM$. It too restricts to produce a chart for TN

$$\begin{array}{ccc} \mathcal{U}_i \times \mathbb{R}^m \cong \mathcal{U}_i \times \mathbb{R}^n \times \mathbb{R}^{m-n} & \longrightarrow & TM \\ \cup & & \cup \\ (\mathcal{U}_i \cap (\mathbb{R}^n \times 0)) \times (\mathbb{R}^n \times 0) & \longrightarrow & TN \end{array}.$$

If we restrict to the complementary summand of $\mathbb{R}^m$, we obtain charts for $\nu(N \subset M)$:

$$\begin{array}{ccc} \mathcal{U}_i \times \mathbb{R}^n \times \mathbb{R}^{m-n} & \longrightarrow & TM \\ \cup & & \cup \\ \left(\mathcal{U}_i \cap (\mathbb{R}^n \times 0)\right) \times (0 \times \mathbb{R}^{m-n}) & \longrightarrow & \nu(N \subset M). \end{array}$$

N sits inside $\nu(N \subset M)$ as the 0-section. //

We shall prove the tubular neighborhood theorem in the special case of an embedding $N \to \mathbb{R}^m$, with N compact.

Let $i : N \hookrightarrow \mathbb{R}^m$ be an embedding. Then

$$\nu_N = \{(x,v) \in N \times \mathbb{R}^m \mid v \perp T_x N\}$$

and we define $f : \nu_N \to \mathbb{R}^m$ by $f(x,v) = i(x) + v$. Clearly $f(x,0) = i(x)$. Now $Df(x,0)$ is invertible, since $Di(x)$ maps onto $T_x N$ and Df maps onto the complement to $T_x N$ since f is linear in v and hence equal to its derivative. Therefore f is a diffeomorphism of some nhood of $(x,0)$ with a nhood of $i(x)$.

Claim: if ε is small enough then f is 1-1 on $\nu(\varepsilon)$, where

$$\nu(\varepsilon) = \{(x,v) \mid \|v\| < \varepsilon\}.$$

Suppose not. Then we can find sequences (x_n, v_n) and (x_n', v_n') with $f(x_n, v_n) = f(x_n', v_n')$ and with $v_n \to 0$ and $v_n' \to 0$. Taking subsequences, we may suppose $x_n \to x_0$ and $x_n' \to x_0'$. But then $ix_0 = f(x_0, 0) = \lim f(x_n, v_n) = \lim f(x_n', v_n') = f(x_0', 0) = ix_0'$, so $x_0 = x_0'$, since i is an embedding. But then, for large n (x_n', v_n') lies in a nhood of $(x_0, 0)$ in which f is 1-1.

Finally $\nu(\varepsilon) \cong \nu_N$ by the diffeomorphism

$$(x,v) \longmapsto \left(x, \frac{v}{1 - \frac{\|v\|^2}{\varepsilon^2}}\right). \qquad //$$