Smooth approximation

The smooth approximation theorem will allow us to prove that a lot of properties of smooth maps hold for continuous maps as well. First, let us recall some point set topological properties of manifolds.

Def: If $\mathcal{V}$ and $\mathcal{U}$ are collections of subsets of M, we say that $\mathcal{V}$ refines $\mathcal{U}$, or $\mathcal{V}$ is subordinate to $\mathcal{U}$, if $\forall V \in \mathcal{V}$ $\exists U \in \mathcal{U}$ such that $V \subset U$.

Def: A collection $\mathcal{V}$ is open if each $V \in \mathcal{V}$ is open. It is locally finite if each $x \in M$ has a nhood $\mathcal{U}$ such that $U \cap V$ is empty for all but a finite number of $V \in \mathcal{V}$.

Def: A topological space M is paracompact if every open covering has a locally finite open refinement which covers.

Recall the following theorems about manifolds, from section PST.

Theorem: A paracompact Hausdorff space is normal.

Theorem: For connected, locally Euclidean, Hausdorff spaces, the following properties are equivalent:

1) second countability
2) metrizability
3) paracompactness
4) σ-compactness
5) Lindelöf

We shall find it useful to have specially tailored atlases to prove the smooth approximation theorem.

<u>Def</u>: A <u>special atlas subordinate to a covering $\mathcal{U}$</u> is an atlas $\mathcal{H} = \{ h_n : D_3(0) \to M \mid n = 1,2,3,\dots \}$ such that

1) $\{ Im(h_n) \}_{n=1}^{\infty}$ is a locally finite open refinement of $\mathcal{U}$,

and 2) $\{ h_n | D_1(0) \}_{n=1}^{\infty}$ is a smooth atlas.

<u>Theorem</u>: Given covers $\mathcal{U}_1, \dots, \mathcal{U}_K$ of a manifold M, there is a special atlas subordinate to all $\mathcal{U}_i$ simultaneously.

<u>Definition</u>: Given a positive function $\delta : M \to \mathbb{R}$ and a metric $d(-,-)$ on N, a <u>δ-homotopy</u> $F : M \times I \to N$ is a continuous map such that

$$d(F(x,t), F(x,0)) < \delta(x) \qquad \forall x \in M \quad \forall t \in I.$$

Thus, a δ-homotopy never moves the image of x more than $\delta(x)$.

<u>Smooth Approximation Theorem</u>: Let $f : M \to N$ be a continuous map, $\delta : M \to \mathbb{R}$ a positive continuous function, d a metric on N. Then there exists a δ-homotopy $F : f \simeq g$ with g smooth. Further, if $A \subset M$ is closed and $f|A$ is smooth, then we can choose F to be constant on A: $F(a,t) = f(a)$ $\forall t$.

<u>Corollary</u>: If M is an m-dimensional manifold and $m < n$ then every continuous map $M \to S^n$ is homotopic to a constant.

In order to prove the Smooth Approximation Theorem, we will introduce a tool which will be useful in many situations.

<u>Def:</u> If $f: M \to \mathbb{R}$, the <u>support of f</u>, <u>supp(f)</u>, is the closure of the set on which f is nonzero:

$$supp(f) = \overline{f^{-1}(\mathbb{R} \setminus 0)}$$

A <u>partition of unity</u> on M is a collection $\{U_i, \phi_i\}$ such that

(1) $\{U_i\}$ is a locally finite open covering of M
(2) $\phi_i : M \to \mathbb{R}$ is a smooth function
(3) $\phi_i(x) \geq 0$ for each x
(4) $\sum_i \phi_i(x) = 1$ for each x

To prove the existence of special atlases we will use the following Lemma.

<u>Lemma:</u> A smooth manifold $M = \bigcup_{i=1}^{\infty} C_i$, each C_i compact and $C_i \subset \overset{\circ}{C}_{i+1}$.

<u>Proof:</u> First suppose M is connected. For each $x \in M$ there is a nhood U_x such that $\overline{U_x}$ is compact. Choose a countable subcover $\{U_{x_1}, U_{x_2}, \cdots\} = \{U_1, U_2, U_3, \ldots\}$. Let $C_1 = \overline{U_1}$, and given C_n, let k_n be an integer such that $k_n > k_{n-1}$ and $(k_1 = 1)$

$$C_n \subset \bigcup_{i=1}^{k_n} U_i$$

(k_n exists because $\{U_i\}$ covers the compact set C_n). Let

$$C_{n+1} = \bigcup_{i=1}^{k_n} \overline{U_i}$$

Then C_{n+1} is compact, being a finite union of compact sets. Clearly

$$\overset{\circ}{C}_{n+1} \supset \bigcup_{i=1}^{k_n} U_i \supset C_n.$$

Since $\{k_n\}$ is strictly increasing, it is clear that $M = \bigcup_{n=1}^{\infty} C_n$.

Now, if M is not connected then it is a countable union of manifolds,

$$M = M_1 \amalg M_2 \amalg M_3 \amalg \cdots$$

since it is second countable. Write each as in the first paragraph:

$$M_n = \bigcup_{i=1}^{\infty} C_{n,i}$$

and define $$C_n = C_{n,1} \amalg C_{n-1,2} \amalg \cdots \amalg C_{1,n}. //$$

We can now prove the theorem on the existence of special atlases.

Proof: First note that our atlas will be subordinate to all of the $\mathcal{U}_i$ if it is subordinate to the single open cover

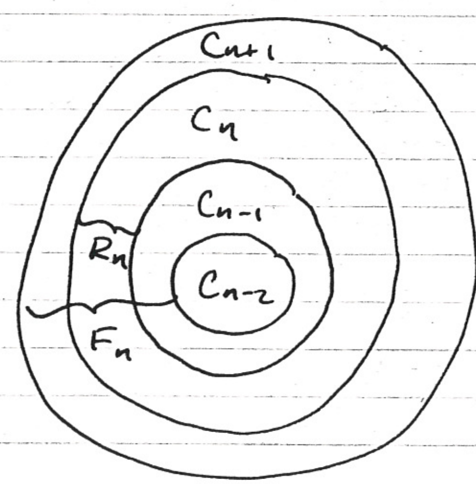
$$\mathcal{U} = \{ U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k} \mid U_{i_j} \in \mathcal{U}_j \}$$

so we may assume we have only a single cover $\mathcal{U} = \{U_i\}$ to consider. Write $M = \bigcup_i C_i$ as in the preceding Lemma and define

$$R_n = C_n - \mathring{C}_{n+1} \quad \text{and} \quad F_n = \mathring{C}_{n+1} - C_{n-2}$$

so that R_n is compact, F_n is open, $R_n \subset F_n$, and $M = \bigcup^{\infty} R_n$. Note that

$$F_n \cap F_m = \phi \quad \text{if } |n-m| > 2.$$



For each $x \in R_n$, let $h_x : D_3(0) \to M$ be a chart with $h_x(0) = x$, $\mathrm{Im}(h_x) \subset F_n$, and $\mathrm{Im}(h_x) \subset U_i$ for some $U_i \in \mathcal{U}$. By compactness of R_n $\exists\, x_1, \ldots, x_k$ such that

$$\{h_{x_i}(D_1(0))\}_{i=1}^{k}$$

covers R_n. Let $\mathcal{H} = \{h_1, h_2, \ldots\}$ be the union of these charts over all n. Then $\mathcal{H}$ is countable, being a countable union of finite sets, $\mathcal{H}$ is subordinate to $\mathcal{U}$, and hence to each of $\mathcal{U}_1, \dots, \mathcal{U}_k$, the collection $\{\mathrm{Im}\, h_i\}_{i=1}^{\infty}$ is locally finite, since the $\{F_n\}$ is, and

$$M = \bigcup_{i=1}^{\infty} h_i(D_1(0)) \quad //$$

<u>Proof of the smooth approximation theorem when $N = \mathbb{R}^n$:</u>

The assumption that f is smooth on A means there is an open set $U \supset A$ such that f is smooth on U. Define two open covers:

$$\mathcal{U}_1 = \{U, M - A\}$$

$$\mathcal{U}_2 = \{ f^{-1}(D_{\varepsilon(x)}(f(x)))\}_{x \in M}$$

and let $\mathcal{H}$ be a special atlas subordinate to $\mathcal{U}_1$ and $\mathcal{U}_2$. This means there are functions (with $\mathbb{N} = \{1,2,3,\dots\}$)

$$\alpha_1 : \mathbb{N} \to \{1,2\} \quad \text{and} \quad \alpha_2 : \mathbb{N} \to M$$

such that
$$\alpha_1(n) = 1 \iff \mathrm{Im}\, h_n \subset U$$

$$\alpha_1(n) = 2 \iff \mathrm{Im}\, h_n \subset M - A$$

and
$$\mathrm{Im}\, h_n \subset f^{-1}\left(D_{\varepsilon(\alpha_2(n))} f(\alpha_2(n))\right).$$

Let $\phi : D_3(0) \to \mathbb{R}$ be a smooth bump function with $\phi | D_1 \equiv 1$ and $\phi | D_3 - D_2 \equiv 0$, and let $\tilde{\Phi}_n : M \to \mathbb{R}$ be

$$\tilde{\Phi}_n(h_n(x)) = \phi(x), \quad \text{and}$$

$$\tilde{\Phi}_n(m) = 0 \qquad \text{if } m \notin \mathrm{Im}\, h_n.$$

Then $\tilde{\Phi}_n$ is smooth, and if we let $\phi_n(x) = \dfrac{\tilde{\Phi}_n(x)}{\sum_i \tilde{\Phi}_i(x)}$, then

$\{\phi_n\}$ is a smooth partition of unity subordinate to $\mathcal{H}$.

Define $f_n : M \to \mathbb{R}^n$ by $\quad f_n(x) = f(x) \quad$ if $\alpha_1(n) = 1$

$$f_n(x) = f(\alpha_2(n)) \quad \text{if } \alpha_1(n) = 2.$$

Then $\phi_n(x) f_n(x)$ is smooth for each n. Define the required homotopy F by

$$F(x,t) = (1-t) f(x) + t \sum_n \phi_n(x) f_n(x).$$

Then $F(x,0) = f(x)$ and $F(x,1)$ is smooth. If $a \in A$ then $\phi_n(a) \neq 0$ only if $\operatorname{Im} h_n \subset \mathcal{U}$, i.e. $\alpha_1(n) = 1$, so $f_n(\ ) = f(a)$ and $F(a,t) = f(a)$. Finally,

$$\begin{aligned}
|F(x,t) - f(x)| &= \Big| (1-t) f(x) + t \sum_n \phi_n(x) f_n(x) - f(x) \Big| \\
&= t \Big| \sum_n \phi_n(x) (f_n(x) - f(x)) \Big| \\
&= t \Big| \sum_{\substack{n \\ \alpha_1(n)=2}} \phi_n(x) (f(\alpha_2(n)) - f(x)) \Big| \\
&\leq t \sum_{\substack{n \\ \alpha_1(n)=2}} \phi_n(x) \, |f(\alpha_2(n)) - f(x)|
\end{aligned}$$

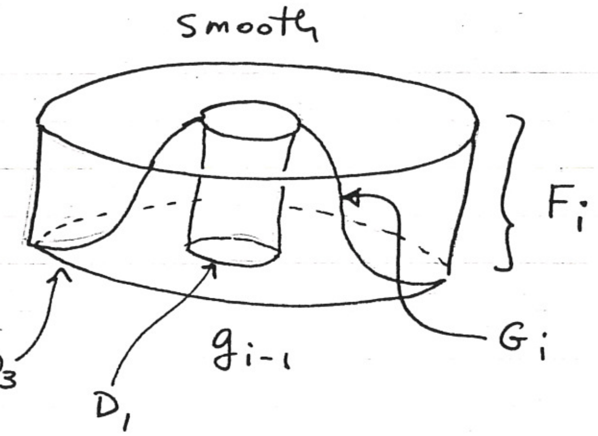
Now, if $\phi_n(x) \neq 0$ then $x \in \operatorname{Im} h_n \subset f^{-1}(D_{\varepsilon(\alpha_2(n))}(f(\alpha_2(n))))$, which implies that $|f(x) - f(\alpha_2(n))| < \varepsilon(\alpha_2(n))$. If this were $\varepsilon(x)$ we would be done. This last step will be filled in later. //

<u>Proof of the general case</u>: We will gradually alter f, making it smooth on the n^{th} chart of a special atlas at step n, using the result for $N = \mathbb{R}^n$.

Choose an atlas $\{k_n : \mathbb{R}^n \to N\}_{n=1}^{\infty}$ for N and let $\mathcal{H}$ be a special atlas subordinate to $\{f^{-1}(\operatorname{Im} k_n)\}$. By renumbering the charts for N we may assume that $f(\operatorname{Im} h_n) \subset \operatorname{Im} k_n$.

Let $g_0 = f$, and, for $i \geq 1$, consider the diagram

$$\begin{tikzcd} D_3(0) \arrow[r, "f_i"] \arrow[d, "h_i"'] & \mathbb{R}^n \arrow[d, "k_i"] \\ M \arrow[r, "g_{i-1}"'] & N \end{tikzcd}$$



where $f_i = k_i^{-1} g_{i-1} h_i$. By the preceding proof there is a homotopy

$$F_i : D_3(0) \times I \longrightarrow \mathbb{R}^n$$

such that

$$F_i(x, 0) = f_i(x)$$

$$F_i(x, 1) \text{ is smooth}$$

$$d\left(k_i F_i(x,t),\, g_{i-1} h_i(x) \right) < \frac{\varepsilon(h_i(x))}{2^i}$$

and

$$F_i(a,t) = f_i(a) \quad \text{if} \quad h_i(a) \in A \cup \bigcup_{j=1}^{i-1} h_j(\bar{D}_1).$$

Let
$$G_i(h_i(x), t) = k_i F_i(x, \phi(x) t)$$

and
$$G_i(m,t) = g_{i-1}(m) \quad \text{if} \quad m \notin \operatorname{Im} h_i,$$

where $\phi(x)$ is the bump function with $\phi|D_1 \equiv 1$, $\phi|D_3 - D_2 \equiv 0$.

① G_i is well defined and continuous since $\phi(x) = 0$ for $x \in D_3 - D_2$ so that $G_i(h_i(x), t) = k_i F_i(x, 0) = g_{i-1} h_i(x)$.

② $G_i(a,t) = g_{i-1}(a)$ if $a \in A \cup \bigcup_{j=1}^{i-1} h_j \bar{D}_1 - \operatorname{Im} h_i$ and

$G_i(h_i a, t) = k_i F_i(a, \phi(x) t) = k_i f_i(a) = g_{i-1} h_i(a)$ if $h_i(a)$ is in $\left(A \cup \bigcup_{j=1}^{i-1} h_j \bar{D}_1\right) \cap \operatorname{Im} h_i$. Thus G_i does not move $A \cup \bigcup_{j=1}^{i-1} h_j \bar{D}_1$.

③ The homotopy G_i starts at g_{i-1} since $k_i F_i(h_i^{-1}(m), 0) = k_i f_i h_i^{-1}(m)$
$= g_{i-1}(m)$.

④ $G_i(h_i(x), 1) = k_i F_i(x, 1)$ if $x \in \overline{D_1}$, and this is smooth, so $G_i|_1$ is smooth on

$$A \cup \bigcup_{j=1}^{i} h_j(\overline{D_1})$$

Thus, G_i makes f smooth on $h_i(\overline{D_1})$ without altering the part which had been smoothed by $G_1, \dots, G_{i-1}$. Thus, we define

$$G(x,t) = \begin{cases} G_1(x, 2t) & 0 \le t \le 1/2 \\ G_2(x, 4t-2) & 1/2 \le t \le 3/4 \\ G_3(x, 8t-6) & 3/4 \le t \le 7/8 \\ \vdots & \vdots \\ G_i(x, 2^i t - (2^{i-1} - 2)) & 1 - 1/2^{i-1} \le t \le 1 - 1/2^i \\ \vdots & \vdots \end{cases}$$

If $x \in h_i(\overline{D_1})$ then $G(x,t)$ is constant in t for $t > 1 - \frac{1}{2^i}$, so $G(x,t)$ can be defined to be continuous at $t = 1$. Similarly, $G(x,t) = G_i(x,t)$ which is smooth in a nhood of x, so G is smooth. Clearly $G(a,t) = f(a)$ for all t. Finally, the distance

$$\begin{aligned} d(f(x), G(x,t)) &\le d(f(x), G_1(x,t)) + d(G_1(x,t), G_2(x,t)) + \cdots \\ &< \frac{\varepsilon(x)}{2} + \frac{\varepsilon(x)}{4} + \cdots \le \varepsilon(x) \quad // \end{aligned}$$

This proof also works to prove a more general version of our earlier transversality theorem.

<u>Thom's Transversality Theorem</u>: Let $f: M \to N$ be smooth, d any metric on N, ε any positive function on M, $Z \subset N$ any submanifold. Then there is a smooth homotopy F which moves each point less than ε, such that $F_1 \pitchfork Z$. If $A \subset M$ is closed and $f|A \pitchfork Z$ then F can be taken to be constant on A.

Proof: Recall that we showed that for $f: M \to \mathbb{R}^n$ and any $\varepsilon > 0$ there is $\beta \in \mathbb{R}^n$ with $|\beta| < \varepsilon$ such that the map

$$f_\beta(m) = f(m) + \beta$$

is transverse to a given submanifold Z. We need only note that there is a smooth homotopy

$$F(m,t) = f(m) + t\beta$$

from f to f_β, and then use the technique of the preceding proof.//

Proof of Cor: Let $f: M \to S^n$, $m < n$. Then $f \simeq g: M \to S^n$ with g smooth. But then g is not onto, since regular values are dense and no point in the image of g is regular. Hence g factors through $S^n - \{pt\} \cong \mathbb{R}^n$ and is therefore homotopic to a constant. Hence f is homotopic to a constant.//

Note: The homotopy groups of spheres $\pi_m S^n$ are the homotopy classes of continuous maps $S^m \to S^n$. We have just shown that

$$\pi_m S^n = 0 \quad \text{if } m < n.$$

We will show later that $\pi_n S^n = \mathbb{Z}$, the integers. In general, the determination of $\pi_{n+i} S^n$ is quite hard. For $i < n$, we have

$$\begin{array}{ll}
\pi_{n+1} S^n = \mathbb{Z}/2 & n > 2 \\
\pi_{n+2} S^n = \mathbb{Z}/2 & n > 3 \\
\pi_{n+3} S^n = \mathbb{Z}/24 & n > 4 \\
\pi_{n+4} S^n = 0 & n > 5 \\
\pi_{n+5} S^n = 0 & n > 6 \\
\pi_{n+6} S^n = \mathbb{Z}/2 & n > 7
\end{array}$$

etc.