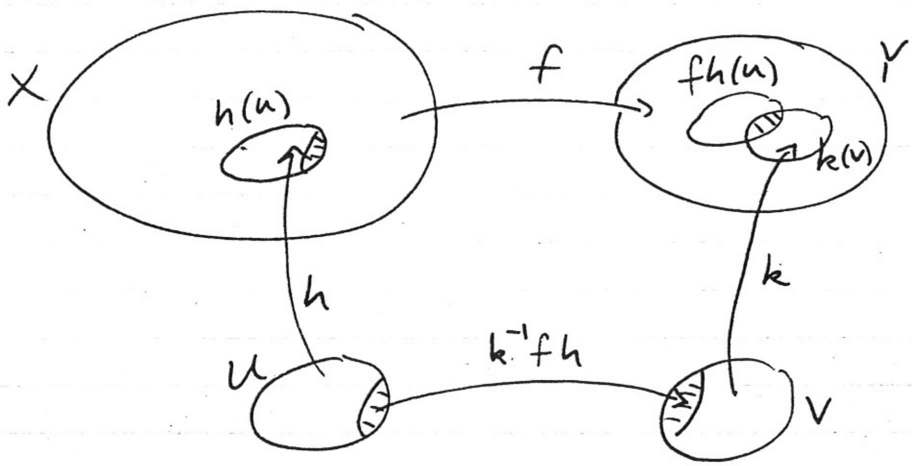
Def: A <u>smooth structure</u> on a topological manifold X is an atlas $\mathcal{H} = \{h_\alpha\}_{\alpha \in A}$ such that $h_{\beta\alpha} = h_\beta^{-1} h_\alpha$ is smooth (i.e. C^∞) for all $\alpha, \beta \in A$.

Def: A <u>smooth map</u> $f : (X, \mathcal{H}) \to (Y, \mathcal{K})$ is a map $f : X \to Y$ such that all possible compositions $k^{-1} f h$ are smooth, $h \in \mathcal{H}$, $k \in \mathcal{K}$.



Def: Smooth structures $\mathcal{H}$ and $\mathcal{H}'$ are <u>equivalent</u> if

$$1_X : (X, \mathcal{H}) \to (X, \mathcal{H}') \quad \text{and} \quad 1_X : (X, \mathcal{H}') \to (X, \mathcal{H})$$

are smooth maps.

It is equivalent to require that $\mathcal{H} \cup \mathcal{H}'$ be a smooth structure. An equivalence class of smooth structures is called a <u>smoothing</u>.

Def: A <u>diffeomorphism</u> is a smooth map with a smooth inverse.

Notes: Poincaré and others started studying this around the end of the 19th century. It was originally thought that

1) every topological manifold would have a smoothing, and 2) smoothings would be unique.

But in the mid 1950's Kervaire constructed a 10-manifold with no smoothing, and Milnor showed that S^7 has 28 distinct smoothings. Also,

$\forall n \ \exists N > n$ such that S^N has a unique smoothing,

and

$\forall n \ \exists N$ such that S^N has more than n smoothings.

Wu-Yi Hsiang and Wu-Chung Hsiang showed that the standard smoothing on S^n is the most symmetrical, in the sense that it has the largest automorphism group.

Also, $\mathbb{R}^n$ has a unique smoothing for $n \neq 4$, but in 1980 Simon Donaldson showed that there are uncountably many (distinct) smooth structures on $\mathbb{R}^4$ by studying the moduli space of solutions to the Yang-Mills equation. Addition and scalar product are smooth functions for only one of these smoothings (the standard one).

<u>Example</u>: The map $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3$ is a homeomorphism but not a diffeomorphism, since $h^{-1}(x) = x^{1/3}$ is not smooth at 0.

<u>Exercise</u>: If $\mathcal{H}$ is a smooth structure on X and $X \xrightarrow{h} X$ is a homeomorphism then there is a smooth structure $\mathcal{K}$ on X such that $h: (X, \mathcal{H}) \to (X, \mathcal{K})$ is a diffeomorphism.

<u>Note</u>: This does not produce an essentially new smooth structure on X, since it is diffeomorphic to the old one by means of h.