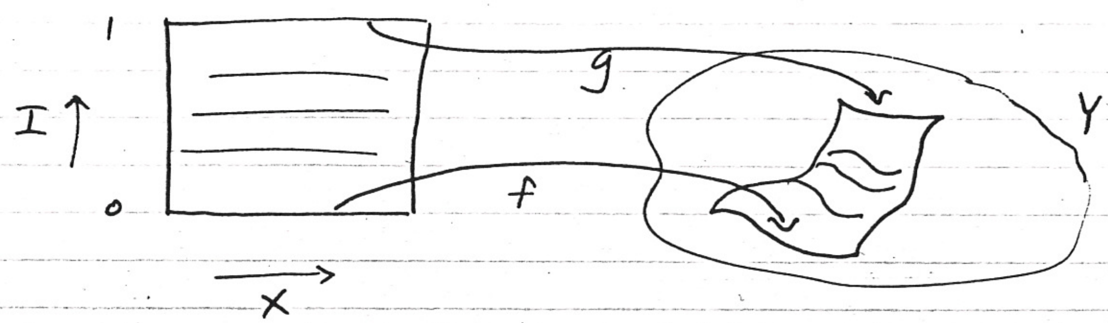
Stability and Transversality

We start with a formalization of the notion of continuous deformation. We shall concentrate primarily on smooth maps here, but in the next section we shall see that the continuous and smooth notions coincide as a consequence of the smooth approximation theorem.

Def: Let $I = [0,1]$, the closed interval $\{x \in \mathbb{R} \mid 0 \le x \le 1\}$. Given topological spaces X and Y, and continuous maps $f, g : X \to Y$ we say <u>f is homotopic to g</u>, written <u>$f \simeq g$</u>, if there exists a continuous function

$$F : X \times I \longrightarrow Y$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We say that <u>F is a homotopy from f to g</u>, written <u>$F : f \simeq g$</u>.



It is convenient to write F_t for the function $X \to Y$, $F_t(x) = F(x,t)$. Thus $F_0 = f$ and $F_1 = g$.

If $f, g : M \to N$ are smooth maps between smooth manifolds, we say $F : M \times I \to N$ is a <u>smooth homotopy</u> from f to g if it is a homotopy $F : f \simeq g$ and there exists $\varepsilon > 0$ and a smooth map $\tilde{F} : M \times (-\varepsilon, 1+\varepsilon) \longrightarrow N$ such that $F = \tilde{F} / M \times I$.

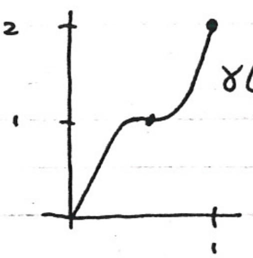
<u>Note</u>: This is the usual strategy for defining smoothness on a closed set C: a function defined on C is smooth if it can be extended to a smooth function on a neighborhood of C.

Lemma: Homotopy and smooth homotopy are equivalence relations.

Proof: The map $F(x,t) = f(x)$ is a homotopy $F : f \simeq f$. If $F : f \simeq g$ then $G(x,t) = F(x, 1-t)$ is a homotopy $G : g \simeq f$. Finally if $F : f \simeq g$ and $G : g \simeq h$ then

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ G(x, 2t-1) & 1/2 \le t \le 1, \end{cases}$$

defines a homotopy $H : f \simeq h$. In the smooth case, we must replace the function $t \mapsto 2t$ by a function $\gamma : [0,1] \to [0,2]$ which is constant on a neighborhood of $1/2$, smooth, and nondecreasing, so that



$$H(x,t) = \begin{cases} F(x, \gamma(t)) & 0 \le t \le 1/2 \\ G(x, \gamma(t)-1) & 1/2 \le t \le 1 \end{cases}$$

will be smooth at $t = 1/2$. //

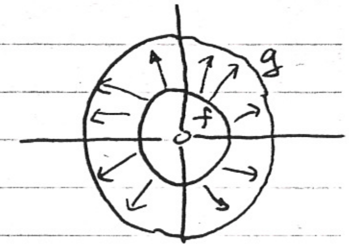
Lemma: If $C \subset \mathbb{R}^n$ is convex then any two (smooth) functions $f, g : M \to C$ are (smoothly) homotopic.

Proof: Let $x_0 \in C$ and let $h : M \to C$ be the constant function $h(x) = x_0$ for all $x \in M$. It suffices to show $f \simeq h$, and the linear homotopy

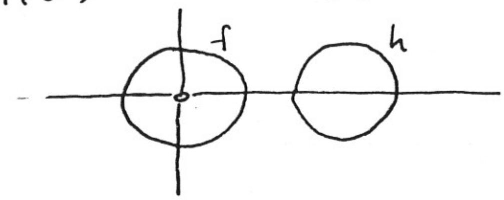
$$F(x,t) = t\, x_0 + (1-t) f(x)$$

is a homotopy $f \simeq h$, smooth if f is smooth. //

Example: Let $f, g : S^1 \to \mathbb{R}^2 - \binom{0}{0}$ be $f(\theta) = \binom{\cos\theta}{\sin\theta}$ and $g(\theta) = \binom{2\cos\theta}{2\sin\theta}$. Then $F(\theta, t) = \binom{(1+t)\cos\theta}{(1+t)\sin\theta}$ is a smooth homotopy $F : f \simeq g$.



Example (cont): Let $h: S^1 \to \mathbb{R}^2 - \binom{0}{0}$ be $h(\theta) = \binom{3+\cos\theta}{\sin\theta}$.



Then $f \not\simeq h$ since any homotopy

$$F: S^1 \times I \longrightarrow \mathbb{R}^2,$$

$F_0 = f$, $F_1 = h$, would have to include $\binom{0}{0}$ in its image. (We will be able to prove this shortly using the Brouwer Fixed Point Theorem.)

Now, we want to say that various properties are preserved under small perturbations. Here is the precise formulation. We will consider only smooth maps here.

Def: A class of maps is stable if, whenever F is a (smooth) homotopy with F_0 in the class, there is an $\varepsilon > 0$ such that F_t is in the class for all $t < \varepsilon$.

Theorem: The following classes of maps from a compact manifold M to any manifold N are stable classes:

1) local diffeomorphisms,
2) immersions,
3) submersions,
4) maps transverse to a given submanifold $Z \subset N$,
5) embeddings,
6) diffeomorphisms.

Proof: Classes (1)-(4) can be written locally as the non-vanishing of an appropriate determinant, hence remain true on an open set in $M \times I$ containing $M \times 0$. Since M is compact, there is an $\varepsilon > 0$ such that $M \times [0, \varepsilon)$ lies in this open set.

For (5) and (6), we need in addition that F_t is 1-1 for all t less than some positive ε. Suppose not. Then there exist x_i, y_i and t_i, $i = 1, 2, 3, \ldots$, such that $t_i \to 0$, $x_i \neq y_i$, and

$F(x_i, t_i) = F(y_i, t_i)$ for each i. Let $G: M \times I \to N \times I$ be $G(x,t) = (F(x,t), t)$. Since M is compact, there is a subsequence such that $(x_i, t_i) \to (x_0, 0)$ and $(y_i, t_i) \to (y_0, 0)$. Then

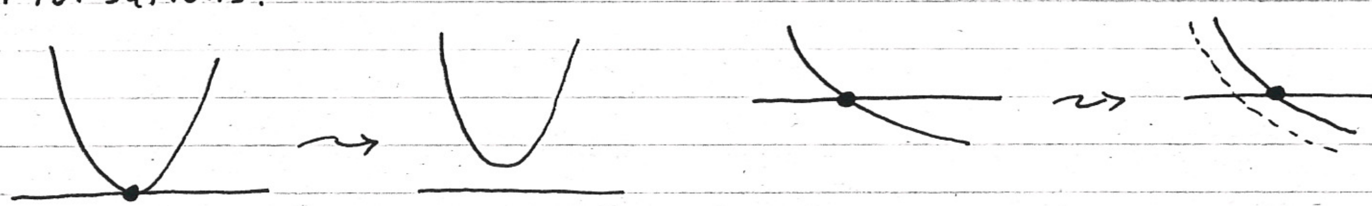
$$G(x_0, 0) = \lim G(x_i, t_i) = \lim G(y_i, t_i) = G(y_0, 0)$$

so $F_0(x_0) = F_0(y_0)$, and hence $x_0 = y_0$. Now, in local coordinates for a nhood of $(x_0, 0)$ we have

$$DG(x_0, 0) = \left(\begin{array}{ccc|c} & DF_0(x_0) & & * \\ \hline 0 & \cdots & 0 & 1 \end{array} \right)$$

which is 1-1, so G is an immersion in a nhood of $(x_0, 0)$. This implies G is 1-1 in a nhood of $(x_0, 0)$, contradicting the assumption that $x_i \neq y_i$. //

This theorem establishes our earlier claim that transverse intersections persist under small perturbations, unlike arbitrary intersections, which may be destroyed by arbitrarily small perturbations.



Non-transverse intersection is not stable

Transverse intersection is stable

We will now use Sard's Theorem to establish a complementary result, namely, that an arbitrary map can be made transverse by an arbitrarily small perturbation.

<u>Theorem</u>: Suppose S is a smooth manifold and $F: M \times S \longrightarrow N$ is a smooth map. For $s \in S$, let $f_s : M \longrightarrow N$ be the map $f_s(x) = F(x,s)$. Suppose $Z \subset N$ is a submanifold. If $F \pitchfork Z$ then $f_s \pitchfork Z$ for almost all $s \in S$.

<u>Proof</u>: Since $F \pitchfork Z$, $W = F^{-1}(Z)$ is a submanifold of $M \times S$. Let $\pi : M \times S \longrightarrow S$ be projection : $\pi(x,s) = s$. We will show that if $s \in S$ is a regular value for $\pi|W$ then $f_s \pitchfork Z$. By Sard's Theorem, almost all values are regular values, so we will be done.

$$\begin{tikzcd} (x,s) \in W \arrow[r] \arrow[d, phantom, "\cap"] & Z \arrow[d, phantom, "\cap"] \\ M \times S \arrow[r, "F"] \arrow[d, "\pi"] & N \\ S & \end{tikzcd}$$

If $f_s(x) = z$ then transversality says that

$$\operatorname{Im} T_{(x,s)} F + T_z Z = T_z N$$

so if $a \in T_z N$ then there exists $(w,u) \in T_x M \times T_s S$ such that

$$b = a - T_{(x,s)} F(w,u) \in T_z Z .$$

$$\begin{tikzcd} T_{(x,s)} W \arrow[r] \arrow[d] & T_z Z \arrow[d] \\ T_x M \times T_s S \arrow[r, "TF"] \arrow[d, "T\pi = \pi"] & T_z N \\ T_s S & \end{tikzcd}$$

Now, if s is a regular value for $\pi|W$ then $T\pi : T_{(x,s)} W \to T_s S$ is onto, but $T\pi(w,u) = u$ so this means there exists $w' \in T_x M$ such that $(w',u) \in T_{(x,s)} W \subset T_{(x,s)} M \times S$. Now $F(W) \subset Z$ so $T_{(x,s)} F(w',u) \in T_z Z$. Therefore

$$\begin{aligned} a &= T_{(x,s)} F(w,u) + b \\ &= T_{(x,s)} F(w-w',0) + T_{(x,s)} F(w',u) + b \\ &= T_x f_s (w-w') + \left(T_{(x,s)} F(w',u) + b\right) \in \operatorname{Im} T_x f_s + T_z Z . // \end{aligned}$$

<u>Corollary</u>: Suppose $f : M \longrightarrow \mathbb{R}^n$ is smooth and $Z \subset \mathbb{R}^n$ is a submanifold. For $s \in \mathbb{R}^n$, let $f_s(x) = f(x) + s$. Then for any $\varepsilon > 0$ there exists $s \in \mathbb{R}^n$ such that $|s| < \varepsilon$ and $f_s \pitchfork Z$.

<u>Proof</u>: Let $S = D_\varepsilon(0) = \{ x \in \mathbb{R}^n \mid |x| < \varepsilon \}$, and let

$F : M \times S \longrightarrow \mathbb{R}^n$ be $F(m, s) = f(m) + s$. Since $T_s S$ maps onto $T_s \mathbb{R}^n$, $F \pitchfork Z$. Hence $f_s \pitchfork Z$ for almost all $s \in S$. //