Stokes Theorem

Thm: M a compact oriented n-manifold,
ω an $(n-1)$ form on M;

$$\Rightarrow \qquad \int_{\partial M} \omega = \int_M d\omega$$

Pf. By linearity of the integral wma ω has compact support in a chart. Thus

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $\widehat{dx_i}$ means dx_i is omitted.

If the chart $\mathcal{U}$ is open in $\mathbb{R}^n$ (i.e. $\mathcal{U}$ has no boundary)

then $\int_{\partial \mathcal{U}} \omega = 0$ and

$$\int_{\mathcal{U}} d\omega = \int_{\mathcal{U}} \sum_{i=1}^{n} (-1)^{i-1} df_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \int_{\mathcal{U}} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n$$

$$= \int_{\mathbb{R}^n} \left(\sum \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n$$

Now, by Fubini's theorem $\int_{\mathbb{R}^n} = \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}}$ and order does not matter. But

$$\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i = \lim_{A \to \infty} \lim_{B \to -\infty} f_i(x_1, \dots, A, \dots, x_n) - f_i(x_1, \dots, B, \dots x_n)$$

$$= 0 - 0 = 0.$$

So the theorem holds.

Now suppose $U \subset \mathbb{R}^n_+$ does have boundary: $U \cap \partial\mathbb{R}^n_+ \neq \phi$.
Then the calculation of $\int_U d\omega$ is the same except when $i = n$. There we get

$$\int_{\mathbb{R}_+} \frac{\partial f_n}{\partial x_n}\, dx_n = \int_0^\infty \frac{\partial f_n}{\partial x_n}\, dx_n = \lim_{A\to\infty} f_n(x_1, \dots, x_{n-1}, A) - f_n(x_1, \dots, x_{n-1}, 0)$$

$$= -f_n(x_1, \dots, x_{n-1}, 0)$$

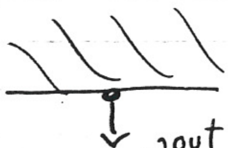
so $$\int_U d\omega = \int_{\mathbb{R}^{n-1}} -f(x_1, \dots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Now $\int_{\partial U} \omega = \int_{\partial\mathbb{R}^n_+} \omega$. Note $dx_n = 0$ on $\partial\mathbb{R}^n_+$, so all the terms in ω except for $i = n$ are 0. Thus

$$\int_{\partial U} \omega = \int_{\partial\mathbb{R}^n_+} (-1)^{n-1} f_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

Now $\mathbb{R}^{n-1} \xrightarrow[\cong]{} \partial\mathbb{R}^n_+$ sends the standard orientation of $\mathbb{R}^{n-1}$ to $(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$ since

$$\nu^{out} \oplus T\partial\mathbb{R}^n_+ = T\mathbb{R}^n_+$$



$\nu^{out} = -e_n$

$$\langle -e_n \rangle \oplus (-1)^n \langle e_1 \wedge \cdots \wedge e_{n-1} \rangle \quad \langle e_1, \dots, e_n \rangle$$

$$= (-1)^{n-1} \langle e_n \wedge e_1 \wedge \cdots \wedge e_{n-1} \rangle$$

$$= \langle e_1 \wedge \cdots \wedge e_n \rangle$$

So $$\int_{\partial u} \omega = \int_{\partial \mathbb{R}^n_+} (-1)^{n-1} f_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} -f_n(x_1, \dots, x_{n-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

$$= \int_u \omega. \quad //$$

Theorem: If
$$\begin{tikzcd} M = \partial W \arrow[r, "f"] \arrow[d, phantom, "\cap"] & N \\ W \arrow[ur, "F"'] & \end{tikzcd}$$
, $f = F|\partial W$, then $\int_M f^*\omega = 0$.

Pf:
$$\begin{tikzcd} f^*\omega & \omega \arrow[l] \arrow[dl] \\ F^*\omega \arrow[u] & \end{tikzcd}$$

so $\int_M f^*\omega = \int_{\partial W} F^*\omega = \int_W d(F^*\omega)$

$= \int_W F^*(d\omega) = 0$

since $d\omega = 0$, because ω is an $n-1$ form on an $n-1$ manifold //

Corollary: If $f_0 \simeq f_1$ then $\int_M f_0^*\omega = \int_M f_1^*\omega$.

Now, suppose ω is a p-form on M and $Z \subset M$ is a p-dim submanifold. Then we may compute

$$\int_Z \omega$$

We start with $p = 1$.

<u>Cor</u>: If M is simply connected then

(1) $\int_\gamma \omega = 0$ for all closed curves γ in M

(2) $\int_\gamma \omega$ depends only on the endpoints of γ

(3) all closed 1-forms are exact.

<u>Recall</u>: ω is a <u>closed</u> p-form if $d\omega = 0$

ω is an <u>exact</u> p-form if $\omega = d\nu$ for some ν.

<u>Pf</u>:
$$\begin{array}{ccc} S^1 & \xrightarrow{\gamma} & M \\ \cap & \nearrow_{\Gamma} & \\ D^2 & & \end{array}$$

$\exists \Gamma$ since M is simply connected, so (1) follows from the Theorem.

Then (1) $\Rightarrow$ (2) since γ_1 followed by γ_2^{-1} is a closed curve if γ_1 and γ_2 have the same endpoints. Finally (2) $\Rightarrow$ (3) by choosing $x_0 \in M$ and letting $f(x) = \int_{x_0}^{x} \omega$. Then $df = \omega$. //

<u>Prop</u>: If Z is a submanifold without boundary then

$$\int_Z \omega = \int_Z \omega + d\theta$$

for all p-forms ω and $(p-1)$ forms θ.

Def: $H^p(M) = \dfrac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}}$

Examples: ① $H^0 M = \mathbb{R}^{\#\text{ of components of } M}$

since a function f is closed iff $df = 0$
iff f is locally constant
iff f is constant on components.

② M simply conn $\Rightarrow H^1(M) = 0$ by Cor, part (3).

③ $H^1(\mathbb{R}^2 - 0) \cong \mathbb{R}$ generated by $\dfrac{x\,dy - y\,dx}{x^2+y^2}$

④ $H^i(M) = 0$ for $i > \dim M$.

⑤ $H^m(M) = \mathbb{R}$ generated by the orientation form
if M is an oriented compact m-manifold
without boundary.

⑥ $H^i(\mathbb{R}^m) = 0, \quad i > 0$

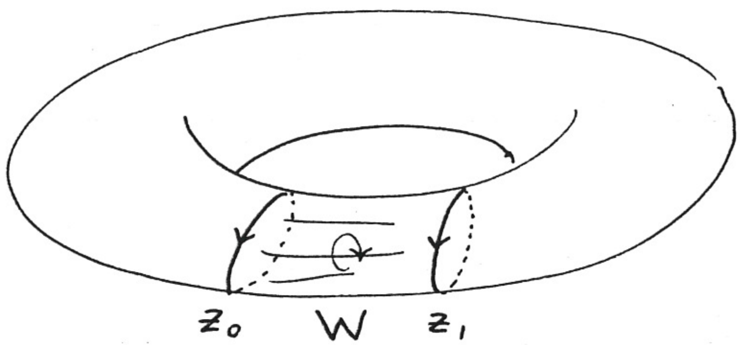
⑦ $$H^i(S^m) = \begin{cases} \mathbb{R} & i = 0, m \\ 0 & \text{otherwise} \end{cases}$$

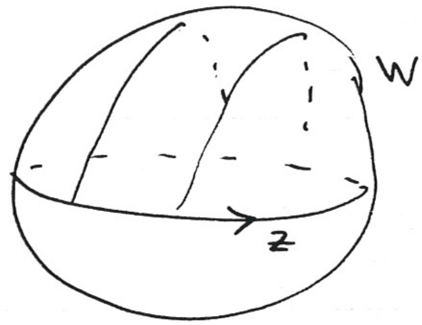
⑧ $$H^i(S^1 \times S^1) = \begin{cases} \mathbb{R} & i = 0 \\ \mathbb{R} \oplus \mathbb{R} & i = 1 \\ \mathbb{R} & i = 2 \\ 0 & i > 2 \end{cases}$$

<u>Theorem</u>: Each p-dim submanifold of M defines a linear functional

$$\int_Z : H^p(M) \to \mathbb{R}$$

<u>Def</u>: Z_0 and Z_1, submanifolds of M, are <u>cobordant</u> if $\exists\ W \subset M$ s.t. $\partial W = Z_1 - Z_0$





$Z_1 \sim Z_0$ $\qquad\qquad$ $Z \sim \phi$

<u>Theorem</u>: $\int_Z$ depends only on the cobordism class of Z.

Thus, if we know $\int_Z \omega$ for one representative from each cobordism class, (the <u>periods</u> of ω) then we can evaluate $\int_Z \omega$ for any Z.

<u>Theorem</u>: $\int_M : H^m(M) \to \mathbb{R}$ is an isomorphism if M is a compact oriented manifold without boundary.