The tangent bundle

The tangent bundle TM of a smooth manifold M provides the most natural context for the derivative of a smooth map. One way to see this is to compare the chain rule expressed in terms of derivatives,

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x)$$

with the form it takes when expressed in terms of the tangent bundle,

$$T(f \circ g) = Tf \circ Tg.$$

There are several ways to define tangents, all of which agree for finite dimensional manifolds. Intuitively, a tangent vector at $x \in M$ should correspond to a speed and direction at which one can pass through x.

Since each $x \in M$ lies in the image of at least one chart $h_\alpha : U_\alpha \to M_\alpha$, we shall use the chart to transport the obvious notion of tangents from Euclidean space to M.

If $x \in U$, an open set in $\mathbb{R}^n$, and $v \in \mathbb{R}^n$, then the curve

$$\gamma(t) = x + tv$$

lies in U for $|t|$ near 0, and $\gamma'(0) = v$. Thus, every $v \in \mathbb{R}^n$ occurs as a tangent vector at x. This means that the space of all tangents to all points of U is $U \times \mathbb{R}^n$: the first coordinate tells the point of tangency, and the second is the tangent there.

A chart $h_\alpha : U_\alpha \to M_\alpha$ is a diffeomorphism, so the space of all tangents to points of M_α is

$$TM_\alpha = M_\alpha \times \mathbb{R}^n$$

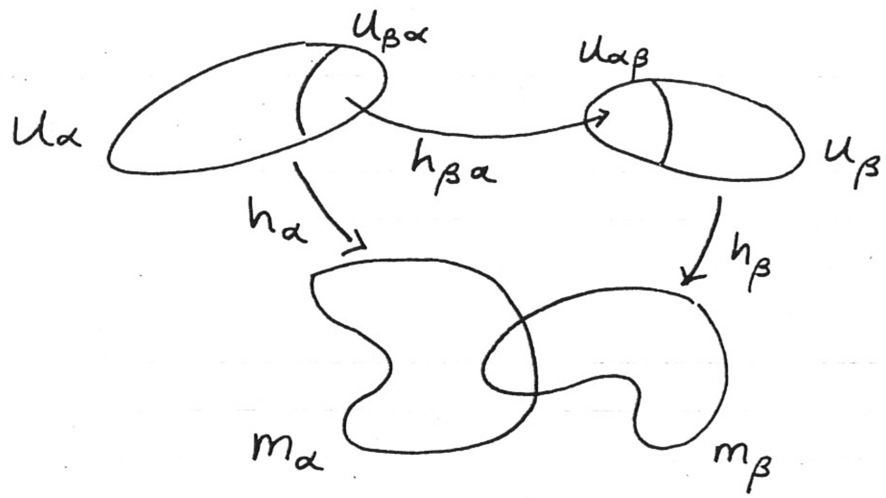
We shall use the transition maps to assemble these local tangent spaces into the single space TM.

Let $(M, \mathcal{H})$ be a smooth manifold with atlas $\mathcal{H} = \{h_\alpha\}_{\alpha \in A}$:

(1) Each $h_\alpha : U_\alpha \to M_\alpha$ is a homeomorphism, where U_α is open in $\mathbb{R}^m$ and M_α is open in M,

(2) $M = \bigcup_\alpha M_\alpha$

(3) Each $h_{\beta\alpha} = h_\beta^{-1} h_\alpha : h_\alpha^{-1}(M_\beta) \to h_\beta^{-1}(M_\alpha)$ is smooth.



$$U_{\beta\alpha} = h_\alpha^{-1}(M_\beta)$$
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We reconstruct M from the open sets $\{U_\alpha\}$ and the transition functions $\{h_{\alpha\beta}\}$ as follows.

Let $X = \coprod_\alpha U_\alpha$ with natural inclusion $i_\alpha : U_\alpha \to X$.

Let $h : X \to M$ by $h i_\alpha = h_\alpha : U_\alpha \to M$. Define $\sim$ on X by

$$i_\alpha(x) = i_\beta(y) \iff y = h_{\beta\alpha}(x)$$

and let $M' = X/\sim$ with quotient map $q : X \to M'$. Since h is constant on equivalence classes it induces $h' : M' \to M$. The atlas $\{h'_\alpha\}$ gives a smooth structure on M' with transition functions $h'_{\beta\alpha} = h_{\beta\alpha}$.

$$\begin{tikzcd}
U_\alpha \arrow[r, "i_\alpha"] \arrow[rr, bend left, "h'_\alpha"] \arrow[dr, "h_\alpha"'] & X \arrow[r, "q"] \arrow[d, "h"] & M' \arrow[dl, "h'"] \\
& M &
\end{tikzcd}$$

Then h' is a diffeomorphism so we identify M and M' by h'.

Definition of the tangent bundle $TM \xrightarrow{\tau} M$:

Let $\widetilde{X} = \coprod_{\alpha \in A} (U_\alpha \times \mathbb{R}^m)$ with natural inclusions $U_\alpha \times \mathbb{R}^m \xrightarrow{\tilde{i}_\alpha} \widetilde{X}$.

Define $\sim$ on $\widetilde{X}$ by

$$\tilde{i}_\alpha(x,u) \sim \tilde{i}_\beta(y,v) \iff \begin{array}{l} y = h_{\beta\alpha}(x) \quad \text{and} \\ v = Dh_{\beta\alpha}(x)(u) \end{array}$$

Let $TM = \widetilde{X}/\sim$ with quotient map $\tilde{q} : \widetilde{X} \to TM$.
The maps $\pi_1 : U_\alpha \times \mathbb{R}^m \to U_\alpha$ induce π, π' and τ in the diagram below.

$$\begin{tikzcd}
U_\alpha \times \mathbb{R}^m \arrow[r, "\tilde{i}_\alpha"] \arrow[d, "\pi_1"'] & \widetilde{X} \arrow[r, "\tilde{q}"] \arrow[d, "\pi"'] & TM \arrow[d, "\pi'"'] \arrow[ddr, bend left, "\tau"] & \\
U_\alpha \arrow[r, "i_\alpha"] \arrow[drrr, "h_\alpha"'] & X \arrow[r, "q"] \arrow[drr, "h"] & M' \arrow[dr, "h'"] & \\
 & & & M
\end{tikzcd}$$

$T(h_\alpha) = \tilde{q}\,\tilde{i}_\alpha$

<u>Theorem</u>: TM is a smooth manifold with atlas $T(\mathcal{H}) = \{T(h_\alpha)\}_{\alpha \in A}$, and τ is a smooth, open map.

<u>Proof</u>: The relations $h_{\alpha\alpha} = 1$, $h_{\gamma\beta} h_{\beta\alpha} = h_{\gamma\alpha}$ and $h_{\alpha\beta}^{-1} = h_{\beta\alpha}$ imply that $\sim$ is an equivalence relation, and that $T(h_\alpha) = \tilde{q}\,\tilde{i}_\alpha$ is 1-1. Thus we can define the transition maps

$$T(h_{\beta\alpha}) = T(h_\beta)^{-1}T(h_\alpha) : U_{\beta\alpha}\times\mathbb{R}^m \longrightarrow U_{\alpha\beta}\times\mathbb{R}^m$$

$$T(h_{\beta\alpha})(x,u) = (h_{\beta\alpha}(x), Dh_{\beta\alpha}(x)(u))$$

Since $h_{\beta\alpha}$ is a diffeomorphism, each $Dh_{\beta\alpha}(x)$ is invertible, and hence each $T(h_{\beta\alpha})$ is also a diffeomorphism. Thus, $T(\mathcal{H})$ is a smooth atlas, if it is an atlas. [See Note 1]

Note that all the maps in the diagram preceding the Theorem are open. This is elementary for all but q and $\tilde{q}$. For $\tilde{q}$ it follows from the fact that $\tilde{q}^{-1}(\tilde{q}(U))$ is the set of all points of $\tilde{X}$ equivalent to a point of U, and this is the union of all $T(h_{\beta\alpha})(U\cap Im(\tilde{i}_\alpha))$, which is clearly open. Similarly for q.

Now $T(h_\alpha)$ is a homeomorphism from $U_\alpha\times\mathbb{R}^m$ to $\tau^{-1}M_\alpha$. Open and continuous follow from the fact that $\tilde{q}$ and $\tilde{i}_\alpha$ are open and continuous. One to one follows from $h_{\alpha\alpha}=1$. To see that it maps onto $\tau^{-1}M_\alpha$, suppose $\tau z = h_\alpha x \in M_\alpha$. Since $\tilde{q}$ is onto, $z=\tilde{q}\tilde{i}_\beta(y,v)$ for some v, y and β. Then

$$h_\alpha x = \tau z = \tau\tilde{q}\,\tilde{i}_\beta(y,v)$$
$$= h_\beta y$$

so $y = h_{\beta\alpha}(x)$. Now

$$z = \tilde{q}\tilde{i}_\beta(y,v) = \tilde{q}\tilde{i}_\alpha(x, Dh_{\alpha\beta}(y)(v))$$

showing that $T(h_\alpha)=\tilde{q}\,\tilde{i}_\alpha$ is onto $\tau^{-1}M_\alpha$.

Finally, it is easy to see that TM is 2^{nd} countable and Hausdorff, since it is covered by the open sets $T(h_\alpha)(U_\alpha\times\mathbb{R}^m)$.

We have seen that τ is open. It is smooth because $h_\alpha^{-1}\tau\,T(h_\alpha) = \pi_1$, which is smooth. //

Note 1: The argument given shows that $T(h_{\beta\alpha})$ is a homeomorphism. To show it is a diffeomorphism, observe that its derivative is

$$D(T(h_{\beta\alpha}))(x,u) = \begin{pmatrix} Dh_{\beta\alpha}(x) & 0 \\ D^2h_{\beta\alpha}(x)(u) & Dh_{\beta\alpha}(x) \end{pmatrix}$$

$$\in \mathrm{Hom}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^m)$$

which is easily seen to be smooth.

Note 2: T is a functor from the category of smooth manifolds to the category of smooth manifolds. This means that if $f: M \to N$ is a smooth map, there is an induced smooth map $Tf: TM \to TN$, which satisfies

$$T(1_M) = 1_{TM} \quad \text{and} \quad T(gf) = T(g)T(f)$$

We start with the requirement that if $U \subset E$, $V \subset F$ are open and $f: U \to V$ is smooth then

$$T(f)(x,u) = (f(x), Df(x)(u))$$

This forces the following

Definition: If $M \xrightarrow{f} N$ is smooth, and $h: U \to M$, $k: V \to N$ are charts then $T(f)T(h)(x,u) = T(k)(k^{-1}fh(x), D(k^{-1}fh)(x)(u))$

$$\begin{tikzcd}
U \times \mathbb{R}^m \arrow[r, "T(h)"] \arrow[d, "\pi_1"'] & TM \arrow[r, "T(f)"] \arrow[d, "\tau"] & TN \arrow[d, "\tau"'] & V \times \mathbb{R}^n \arrow[l, "T(k)"'] \arrow[d] \\
U \arrow[r, "h"] & M \arrow[r, "f"] & N & V \arrow[l, "k"']
\end{tikzcd}$$

Theorem: $T(f)$ is a well defined smooth map, and

$$T(1_M) = 1_{TM}, \quad \text{and} \quad T(gf) = T(g)T(f).$$

Proof: Easy. //

Definition: If $x \in M$, the tangent space to M at x, is

$$T_x(M) = \tau^{-1}(x) \subset TM$$

Proposition: $T_x(M)$ is a vector space and if $M \xrightarrow{f} N$ is smooth, then $T_x(f) : T_x(M) \longrightarrow T_x(N)$ is linear.

Proof: The maps

$$\begin{tikzcd} \{a\} \times \mathbb{R}^m \arrow[r, "Th_\alpha"] \arrow[d] & TM \arrow[d] \\ \{a\} \arrow[r, "h_\alpha"] & M \end{tikzcd}$$

, where $h_\alpha(a) = x$,

give $T_x M$ a vector space structure which is independent of α since $D h_{\beta\alpha}(a)$ is a linear isomorphism. Clearly $T(f)$ is linear in the tangent coordinate. //

Definition: $M \xrightarrow{f} N$ is

(1) an immersion if $T_x f$ is 1-1 $\forall x \in M$

(2) a submersion if $T_x f$ is onto $\forall x \in M$

(3) an embedding if it is an immersion and a homeomorphism onto its image.

Examples: (1) $\tau: TM \to M$ is a submersion

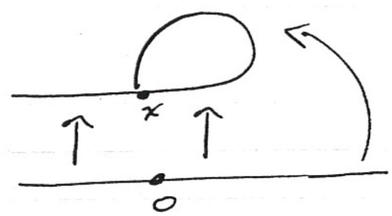
(2) The zero-section $s: M \to TM$, $s(m) = (m, 0)$, is an immersion. Note that s is well defined since every linear map sends 0 to 0.

(3) $\exp: \mathbb{R} \to S^1$ is an immersion and a submersion, as is the natural quotient map $p: S^n \to \mathbb{RP}^n$.

(4) $\exp: \mathbb{R} \to \mathbb{C}^\times$, $\exp(x) = e^{ix}$, is an immersion.

The maps in (3) and (4) show that an immersion may fail to be an embedding by failing to be 1-1. Note that, by the Inverse Function Theorem, an immersion will be locally 1-1, i.e. 1-1 in a neighborhood of each point. More subtle failures occur in (5) and (6)

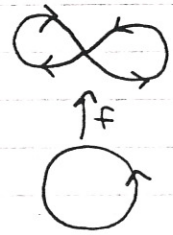
(5) Map $\mathbb{R} \xrightarrow{f} \mathbb{R}^2$ as indicated:



This is a 1-1 immersion, but not a homeomorphism onto its image, because every neighborhood of x has inverse image a neighborhood of 0 together with a set of the form (a, ∞). Or, $f[1, \infty)$ is not closed.

(6) Let $f: \mathbb{R} \to S^1 \times S^1$ be $f(x) = (e^{ix}, e^{i\alpha x})$ where α is irrational. Then f is an immersion, but its image is dense in $S^1 \times S^1$, and is not homeomorphic to $\mathbb{R}$.

(7) Let $f: S^1 \to \mathbb{R}^2$ have image a "figure eight". Then f is an immersion but not an embedding.



The Inverse Function Theorem has the following immediate corollary.

Theorem: If $f: M \to N$ is smooth and $T_m f$ is invertible for some $m \in M$, then there exist neighborhoods $\mathcal{U}$ and V of m and $f(m)$, respectively, such that $f|\mathcal{U}$ is a diffeomorphism $\mathcal{U} \to V$.

Proof: Restrict to charts in neighborhoods of m and $f(m)$, and apply the Inverse Function Theorem there.

Corollary: A 1-1 onto immersion is a diffeomorphism.

Proof: We need only check that its inverse is smooth, and this is ~~a~~ local question which is answered by the Theorem.

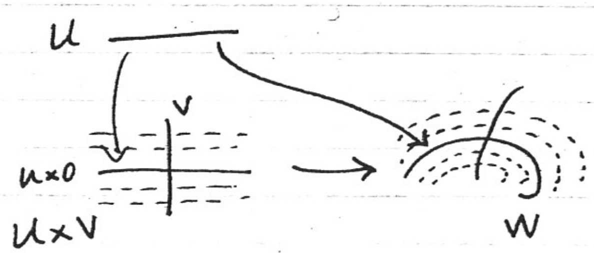
Theorem: An immersion is locally an embedding. More generally, if $f: M \to N$ is smooth and $T_m f$ is 1-1 for some $m \in M$, then f is 1-1 in some neighborhood of m.

Proof: This is an immediate consequence of the Local Straightening Theorem, since the rank of Tf must be constant in a nhood of m because it equals its maximum possible value, $\dim M$, at m. //

A direct proof from the inverse function theorem can be given as follows. (In fact we prove a version of the Local Straightening Theorem.)

Lemma: If $T_m f$ is 1-1 then there exist neighborhoods $\mathcal{U}$ of m, V of 0 in $\mathbb{R}^{n-m}$, and W of $f(m)$, and a diffeomorphism $\ell: \mathcal{U} \times V \to W$ such that $\ell(x, 0) = f(x)$.

$$\begin{tikzcd} \mathcal{U}\times 0 \cong \mathcal{U} \arrow[d] \arrow[dr, "f"] & \\ \mathcal{U}\times V \arrow[r, "\ell", "\cong"'] & W \end{tikzcd}$$

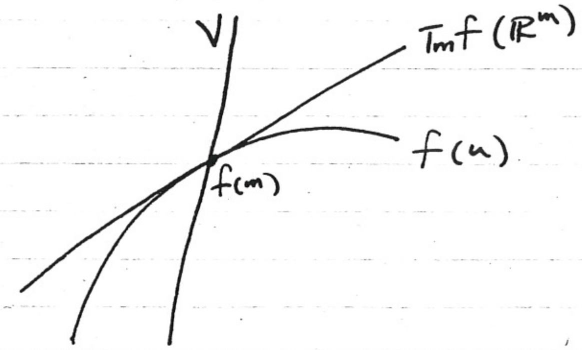


Proof: Since this is a local question, we may restrict to charts about m and $f(m)$, and therefore may assume $f: \mathcal{U} \to W$, $T_m f$ 1-1, $\mathcal{U} \subset \mathbb{R}^m$, $W \subset \mathbb{R}^n$.

Let V be a subspace of $\mathbb{R}^n$ complementary to $T_m f(\mathbb{R}^m)$, so that

$$\mathbb{R}^n \cong T_m f\,(\mathbb{R}^n) \oplus V \cong \mathbb{R}^m \oplus \mathbb{R}^{n-m}.$$

Then the map $\bar{f}: \mathcal{U} \times V \longrightarrow \mathbb{R}^m$, $\bar{f}(u,v) = f(u) + v$, has



$T_{(m,0)}\bar{f} = \begin{pmatrix} T_m f & I \end{pmatrix}$, which is invertible. Therefore, $\bar{f}$ is a diffeomorphism of a neighborhood of $(m,0)$ to a neighborhood of $f(m)$. //

Here is a general result about immersions and embeddings.

Theorem: A 1-1 immersion which is either open or closed as a map to its image is a diffeomorphism.

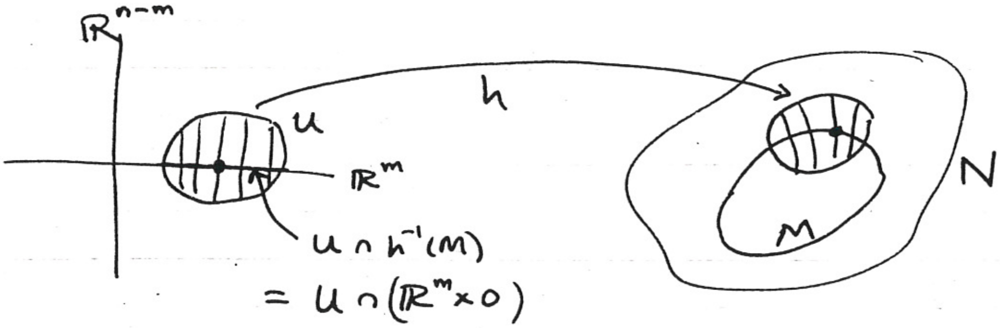
Proof: The inverse is then continuous, so the map is a homeomorphism to its image.

Corollary: If M is compact, then a 1-1 immersion $M \xrightarrow{f} N$ is an embedding.

Proof: Closed in M implies compact, f preserves compactness, and compact in a Hausdorff space implies closed. //

Def: $M \subset N$ is a <u>submanifold</u> if for each $m \in M$ there is a chart of N, $h: U \to N$, with $m \in h(U)$, such that

$$(*) \qquad U \cap h^{-1}(M) = U \cap (\mathbb{R}^m \times 0), \qquad U \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}.$$



Example: $S^n \subset \mathbb{R}^{n+1}$ is a submanifold, since

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_0 + \sqrt{1 - \sum_1^n x_i^2} \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a chart for $\mathbb{R}^{n+1}$ whose restriction to $0 \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ is the standard chart

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} \sqrt{1 - \sum x_i^2} \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{of } S^n.$$

Prop: If M is a submanifold of N then M is a manifold with charts

$$\left\{ U \cap (\mathbb{R}^m \times 0) \xrightarrow{h} h(U) \cap M \;\middle|\; h \text{ satisfies } (*) \right\}$$

Def: The <u>codimension</u> of M in N is $\dim N - \dim M$.

Prop: The image of an embedding is a submanifold.