Tensors and Differential Forms

Def: The <u>tensor product</u> $V \otimes W$ of real vector spaces V and W is the quotient

$$F(V \times W) / I$$

where $F(V \times W)$ is the vector space with basis $V \times W$ and I is the subspace generated by the set

$$(*) \qquad \begin{aligned} &(v_1 + v_2, w) - (v_1, w) - (v_2, w) \\ &(v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ &(cv, dw) - cd\,(v, w) \end{aligned}$$

for all $c, d \in \mathbb{R}$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$.

Write $v \otimes w$ for the coset $(v,w) + I$ of (v,w).

This defines a function $V \times W \xrightarrow{\phi} V \otimes W$ which satisfies

<u>The Universal Mapping Property</u>

There is a 1-1 correspondence between linear transformations $\tilde{B} : V \otimes W \to U$ and bilinear functions $B : V \times W \to U$ via

$$\begin{tikzcd} V \times W \arrow[d, "\phi"'] \arrow[r, "B"] & U \\ V \otimes W \arrow[ur, "\tilde{B}"'] & \end{tikzcd}$$

$$B = \tilde{B}\,\phi.$$

(Since $\tilde{B}$ is linear, it is determined by its values on the basis $\phi(V \times W)$, and bilinearity of B is exactly the condition that $\tilde{B}$ is 0 on I, and hence well defined on $V \otimes W$.)

Further $V \otimes W$ is the only vector space with this Universal Mapping Property.

Properties: (1) $V \otimes W \cong W \otimes V$ by $v \otimes w \leftrightarrow w \otimes v$

(2) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ by $(v \otimes w) \otimes u \leftrightarrow v \otimes (w \otimes u)$

(3) $V^* \otimes W \cong Hom(V, W)$ by $(\alpha \otimes w)(v) = \alpha(v) w$

(4) If $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^m$ are bases for V and W then $\{e_i \otimes f_j\}_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ is a basis for $V \otimes W$.

Property (3) uses the "non-singular pairing"

$$V^* \otimes V \xrightarrow{eval} \mathbb{R}$$

$$\alpha \otimes v \longmapsto \alpha(v)$$

Def: A pairing of V and W is a bilinear function $V \times W \to \mathbb{R}$, or equivalently, a homomorphism

$$\langle \, , \, \rangle : V \otimes W \longrightarrow \mathbb{R}$$

The pairing is non-singular if whenever $w \in W$ is nonzero, $\exists\, v \in V$ such that $\langle v, w \rangle \neq 0$, and whenever $v \in V$ is nonzero, $\exists\, w \in W$ such that $\langle v, w \rangle \neq 0$.

Prop: If $\langle \, , \, \rangle$ is a nonsingular pairing then $V \xrightarrow{\phi} W^*$, by $\phi(v)(w) = \langle v, w \rangle$, is an isomorphism.

Pf: ϕ is 1-1 since $\langle \, , \, \rangle$ is non-singular. Similarly, $\psi : W \to V^*$ by $\psi(w)(v) = \langle v, w \rangle$ is 1-1. Hence $\dim V = \dim W$ and ϕ and ψ are isomorphisms. //

Note: In this case, we have

$$\begin{tikzcd} V \otimes W \arrow[r, "\langle \, , \, \rangle"] \arrow[d, "\phi \otimes 1"'] & \mathbb{R} \\ W^* \otimes W \arrow[ur, "eval"'] & \end{tikzcd}$$

Given a basis $\{e_1, \dots, e_n\}$ of V we have a <u>dual basis</u> $\{e_1^*, \dots, e_n^*\}$ defined by

$$\langle e_i^*, e_j \rangle = \delta_{ij}$$

Similarly, the basis $\{e_1^*, \dots, e_n^*\}$ determines the basis $\{e_1, \dots, e_n\}$.

<u>Def</u>: The <u>(r,s)-tensors</u> on V are the elements of

$$T^{r,s}(V) = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}$$

The <u>(r,s)-tensor bundle on M</u> is the space

$$T^{r,s}(M) = \bigcup_{m \in M} T^{r,s}(T_m M)$$

together with the projection $\tau_{r,s} : T^{r,s}(M) \to M$.

<u>Examples</u> ① Write $T^*M = T^{0,1}(M)$. Given a function $M \xrightarrow{f} \mathbb{R}$ the <u>differential of f</u>, $df: M \to T^*M$, is defined by letting

$$df(m) : T_m M \to \mathbb{R}$$

be the linear map induced by f:

$$\begin{tikzcd} T_m M \subset TM \arrow[r, "Tf"'] \arrow[rr, bend left, "df(m)"] \arrow[d, "\tau"] & T\mathbb{R} \arrow[r] \arrow[d, "\tau = \pi_1"] & \mathbb{R} \\ M \arrow[r, "f"] & \mathbb{R} & \end{tikzcd}$$

Alternatively, $df(v)$ is the derivative of f in the direction v.

② A Riemannian metric is a bilinear function on $T_m M \times T_m M$ for each m which can be considered a $(0,2)$-tensor

$$g : M \to T^{0,2} M$$

under the isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$ with

$V = W = T_m M$. The isomorphism $V^* \otimes W^* \xrightarrow{\phi} (V \otimes W)^*$ is defined by

$$\phi(\alpha \otimes \beta)(v \otimes w) = \alpha(v)\beta(w).$$

Def: The tensor algebra $T(V)$ on V is the vector space

$$T(V) = \bigoplus_{k \geq 0} T^{k,0}(V) = \bigoplus_{k \geq 0} \underbrace{V \otimes \cdots \otimes V}_{k}$$

with the product $(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m)$

$$= v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m.$$

Let $I(V)$ be the ideal in $T(V)$ generated by the set $\{ v \otimes v \mid v \in V \}$. The exterior algebra $\Lambda(V) = T(V)/I$.

If $I_k(V) = I(V) \cap T^{k,0}(V)$ then

$$\Lambda(V) = \bigoplus \Lambda_k(V)$$

where $\Lambda_k(V) = T^{k,0}(V)/I_k(V)$. Write the induced product on $\Lambda(V)$ by $\wedge$, called the wedge product or exterior product. Thus, the equivalence class of $v_1 \otimes \cdots \otimes v_k$ is $v_1 \wedge \cdots \wedge v_k$.

Prop: $\Lambda(V)$ is a graded anti-commutative algebra of dimension 2^d, $d = \dim V$. The dimension of $\Lambda_k(V)$ is $\binom{d}{k}$. If $\{v_1, \ldots, v_d\}$ is a basis of V then

$$\{ v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < \cdots < i_k \}$$

is a basis of $\Lambda_k(V)$. $\Lambda_0(V) \cong \mathbb{R} \cong \Lambda_d(V)$ and $\Lambda_1(V) \cong V$.

Anti-commutative means

if $v \in \Lambda_k(V)$, $w \in \Lambda_\ell(V)$ then $v \wedge w = (-1)^{k\ell} w \wedge v$.

<u>Proof</u>: For $v_1, v_2 \in V$, $(v_1 + v_2) \otimes (v_1 + v_2) \in I$, and

$$(v_1 + v_2) \otimes (v_1 + v_2) = v_1 \otimes v_1 + v_1 \otimes v_2 + v_2 \otimes v_1 + v_2 \otimes v_2$$

so $v_1 \otimes v_2 + v_2 \otimes v_1 \in I$ also. Therefore

$$\begin{aligned} v_1 \wedge v_2 &= v_1 \otimes v_2 + I \\ &= v_1 \otimes v_2 - (v_1 \otimes v_2 + v_2 \otimes v_1) + I \\ &= -v_2 \otimes v_1 + I \\ &= -v_2 \wedge v_1 . \end{aligned}$$

If $v \in \Lambda_k(V)$ then v is a sum of elements $v_{i_1} \wedge \cdots \wedge v_{i_k}$ and it follows that $v \wedge w = (-1)^{k\ell} w \wedge v$ if $w \in \Lambda_\ell(V)$. Therefore the $v_{i_1} \wedge \cdots \wedge v_{i_k}$ with $i_1 < \cdots < i_k$ span $\Lambda_k(V)$, or in other words the $\{ v_\Phi \mid \Phi \subset \{1, \ldots, d\}\}$ span $\Lambda(V)$, where $v_\Phi = v_{i_1} \wedge \cdots \wedge v_{i_k}$ if $\Phi = \{i_1, i_2, \ldots, i_k\}$, $i_1 < \cdots < i_k$.

Now <u>Lemma</u>: $v_1 \otimes \cdots \otimes v_d \notin I$

<u>Pf</u>: Exercise.

Therefore $v_1 \wedge \cdots \wedge v_d \in \Lambda_d(V)$ is nonzero. Now if $\sum a_\Phi v_\Phi = 0$ we see inductively that each $a_\Phi = 0$ ~~so~~ by multiplying by v_Ψ where $\Psi = \{1, 2, \ldots, d\} - \Phi$. It follows that

$$\dim \Lambda_k(V) = \#\text{ of } k \text{ element subsets of } \{1, \dots, d\}$$
$$= \binom{d}{k}$$

and
$$\dim \Lambda(V) = \#\text{ subsets of } \{1, \dots, d\} = 2^d.$$

Finally, $\Lambda_0(V) = \mathbb{R}$ spanned by $v_\emptyset = 1$, $\Lambda_d(V) = \mathbb{R}$ spanned by $v_1 \wedge \cdots \wedge v_d$, and $\Lambda_1(V) = T^{1,0}(V) = V$. //

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<u>Def</u>: A multilinear map $V \times \cdots \times V \xrightarrow{f} W$ is <u>alternating</u> if
$$f(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = \text{sign}(\sigma)\, f(v_1, \dots, v_r)$$
where $\text{sign}(\sigma)$ is the sign of the permutation σ.

Let $A_r(V)$ be the vector space of alternating multilinear functions $V \times \cdots \times V \longrightarrow \mathbb{R}$,

$A_0(V) = \mathbb{R}$, and $A(R) = \bigoplus_r A_r(V)$.

<u>The Universal Mapping Property</u> $A(V) = \Lambda(V)^*$. That is, there is a 1-1 correspondence between alternating multilinear functions f and linear transformations $\bar{f}$ via
$$f = \bar{f}\,\psi\,\phi.$$

$$\begin{tikzcd}
V \times \cdots \times V \arrow[d, "\phi"'] \arrow[r, "f"] & \mathbb{R} \\
V \otimes \cdots \otimes V \arrow[d, "\psi"'] \arrow[ur, dashed, "\tilde{f}"] & \\
\Lambda_k(V) \arrow[uur, dashed, "\bar{f}"'] &
\end{tikzcd}$$

Similarly with $\mathbb{R}$ replaced by any vector space W.

<u>Note</u> : The pairing $\Lambda_k(V^*) \otimes \Lambda_k(V) \longrightarrow \mathbb{R}$

$$(\star) \qquad \langle v_1^* \otimes \cdots \otimes v_k^*, u_1 \otimes \cdots \otimes u_k \rangle = \det(v_i^*(u_j))$$

(and extend by linearity) is nonsingular and defines an isomorphism

$$\Lambda_k(V^*) \cong \Lambda_k(V)^* .$$

Therefore
$$A(V) = \bigoplus_{k=0}^{d} A_k(V) = \bigoplus_{k=0}^{d} \Lambda_k(V)^* = \bigoplus_{k=0}^{d} \Lambda_k(V^*) = \Lambda(V^*)$$

Or, briefly

$$A(V) = \Lambda(V)^* \underset{\alpha}{\cong} \Lambda(V^*)$$

The first isomorphism is the natural one derived from the universal mapping property. The second depends on the pairing $\star$).

There is an alternative to ($\star$), namely

$$(\star\star) \qquad \langle v_1^* \wedge \cdots \wedge v_r^*, u_1 \wedge \cdots \wedge u_r \rangle = \frac{1}{r!} \det(v_i^*(u_j))$$

which yields a second isomorphism $\Lambda(V)^* \underset{\beta}{\cong} \Lambda(V^*)$.

Now $\Lambda(V^*)$ is an algebra under wedge product, and $A(V)$ obtains two different algebra structures from $\Lambda(V^*)$ using the isomorphisms α and β. Write $f \wedge_\alpha g$ and $f \wedge_\beta g$.

We find that if $f \in A_p(V)$ and $g \in A_q(V)$ then

$$f \wedge_\alpha g\,(v_1, \dots, v_{p+q}) = \sum_{\substack{(p,q)\\ \text{shuffles}\\ \pi}} \operatorname{sign}(\pi)\, f(v_{\pi(1)}, \dots, v_{\pi(p)})\, g(v_{\pi(p+1)}, \dots, v_{\pi(p+q)})$$

while

$$f \wedge_\beta g\,(v_1, \dots, v_{p+q}) = \frac{\sum_{\pi \in \Sigma_{p+q}} \operatorname{sign}(\pi)\, f(v_{\pi(1)}, \dots, v_{\pi(p)})\, g(v_{\pi(p+1)}, \dots, v_{\pi(p+q)})}{(p+q)!}$$

where a (p,q) shuffle is a permutation satisfying $\pi(1) < \dots < \pi(p)$ and $\pi(p+1) < \dots < \pi(p+q)$. Therefore

$$f \wedge_\alpha g = \frac{(p+q)!}{p!\,q!}\, f \wedge_\beta g$$

The advantage of $\wedge_\alpha$ is that fewer extra constants enter the formulas, such as the two formulas above. For example, if $f, g \in A_1(V)$ and $v, w \in V$ then

$$f \wedge_\alpha g\,(v, w) = f(v)g(w) - f(w)g(v)$$

while $f \wedge_\beta g\,(v, w) = \frac{1}{2}\big(f(v)g(w) - f(w)g(v)\big)$

and if $f \in A_2$, $g \in A_1$ then

$$f \wedge_\alpha g\,(v_1, v_2, v_3) = f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1)$$

$$f \wedge_\beta g\,(v_1, v_2, v_3) = \frac{1}{6}\big(f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1) - f(v_2, v_1)g(v_3) + f(v_3, v_1)g(v_2) - f(v_3, v_2)g(v_1) \big).$$

Def: The <u>exterior k-form bundle</u> on a manifold M is

$$\begin{array}{c} A_k(M) = \Lambda_k^*(M) = \bigcup_{m \in M} \Lambda_k(T_m M^*) \\ \Big\downarrow \tau_k \\ M \end{array}$$

The manifold structure on $\Lambda_k^*(M)$ is derived from that of M as follows. Let $h : \mathbb{R}^n \to U \subset M$ be a chart with coordinate functions

$$\underbrace{U \xrightarrow{h^{-1}} \mathbb{R}^n \longrightarrow \mathbb{R}}_{x_i}$$

Thus $\{\partial/\partial x_i\}_{i=1}^n$ is a basis for $T_m M$ for each $m \in U$.

The differentials dx_i are the dual basis for $T_m M^*$.

Then

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^{\binom{n}{k}} &\longrightarrow \tau_k^{-1}(U) \\ (x, (r_{i_1 \dots i_k})) &\longmapsto \sum r_{i_1 \dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &\qquad \text{at } h(x) \end{aligned}$$

defines charts for $A_k(M)$.

Def: A <u>differential k-form</u> on M is a cross-section $M \xrightarrow{\omega} A_k(M)$

Thus $\omega(m) \in \Lambda_k(T_m M^*)$ for each $m \in M$, and ω is smooth iff for every k-tuple of tangent vector fields $v_1, \dots, v_k$, the function $M \to \mathbb{R}$ by

$$m \mapsto \omega(m)(v_1(m), \dots, v_k(m))$$

is smooth.

Given any bundle $E \xrightarrow{\pi} M$, the space of (smooth) sections

$$\Gamma(E) = \{ s : M \to E \mid \pi s = 1_M \}$$

If the bundle is trivial, $M \times F \xrightarrow{\pi} M$, $\Gamma(M \times F) = C^\infty(M, F)$.

Since $A_0(M) = M \times \mathbb{R}$, $\Gamma(A_0(M)) = C^\infty(M, \mathbb{R})$. The differential is therefore a linear function

$$d : \Gamma(A_0(M)) \longrightarrow \Gamma(A_1(M))$$
$$f \longmapsto df \, .$$

<u>Theorem</u>: There is a unique linear function $d : \Gamma(A_k(M)) \to \Gamma(A_{k+1}(M))$ such that (1) when $k=0$, d takes the differential of f,

(2) $d^2 = 0$

(3) $d(u \wedge v) = d(u) \wedge v + (-1)^p u \wedge d(v)$ if $u \in \Gamma(A_p(M))$.

<u>Proof</u>: In local coordinates $\omega \in \Gamma(A_k(M))$ can be written

$$\omega = \sum_{\substack{\Phi \subset \{1, \dots, m\} \\ |\Phi| = k}} a_\Phi \, dx_\Phi$$

where $dx_\Phi = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ if $\Phi = \{i_1, \dots, i_k\}$, $i_1 < \dots < i_k$.

Define

$$d\omega = \sum da_\Phi \wedge dx_\Phi \, .$$

Clearly d is linear and satisfies (1). Equality of mixed partials and anticommutativity ($dx_1 \wedge dx_2 = - dx_2 \wedge dx_1$) imply (2).

For (3) suppose that

$u = a\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $v = b\, dx_{j_1} \wedge \cdots \wedge dx_{j_q}$, $p+q=m$.

Then

$$
\begin{aligned}
d(u \wedge v) &= d(ab\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&= ((da)\, b + a\, db) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\
&= (da \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (b\, dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&\quad + (-1)^p (a \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (db \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&= du \wedge v + (-1)^p (u \wedge dv). \quad //
\end{aligned}
$$