Vector Fields and Parallelizable Manifolds

Let M be a smooth manifold with tangent bundle $TM \xrightarrow{\tau} M$.

Def: A vector field on M is a smooth function $X: M \longrightarrow TM$ such that $\tau X = 1_M$.

$$\begin{tikzcd} TM \arrow[d, "\tau"'] \\ M \arrow[u, bend right, "X"'] \end{tikzcd}$$

Thus $X(m) \in T_m M$ is a tangent vector at m.

Examples: ① The zero section $s(m) = 0_m$ is a vector field.

② $\frac{\partial}{\partial x_i} : \mathbb{R}^n \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

is the vector field $\frac{\partial}{\partial x_i}(x) = (x, e_i)$

where e_i is the i^{th} standard basis vector.

③ In local coordinates $h: U \longrightarrow M$, a vector field X can be

$$(Th)^{-1} X(h(x)) = (x, v(x))$$

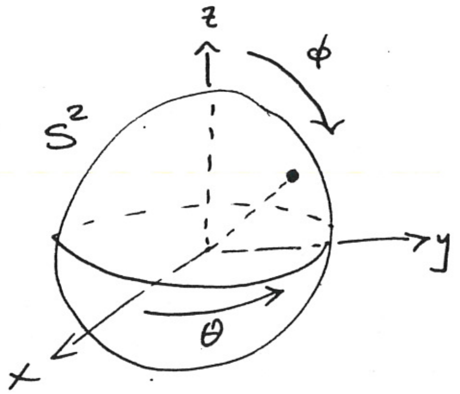
$$\begin{tikzcd} U \times \mathbb{R}^m \arrow[r, "Th"] \arrow[d, "\pi_1"'] & TM \arrow[d, "\tau"'] \\ U \arrow[r, "h"] \arrow[u, bend left, "(1,v)"] & M \arrow[u, bend right, "X"'] \end{tikzcd}$$

where $v: U \longrightarrow \mathbb{R}^m$ is a smooth function. We can write

$$v(x) = \sum_{i=1}^{m} X_i(x) \frac{\partial}{\partial x_i}.$$

This is the expression of X in terms of the local coordinates $(x_1, \dots, x_n) \in U$.

④ Let $h: \mathbb{R}^2 \longrightarrow S^2$ be the stereographic projection chart $h\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{r^2+1}\begin{pmatrix} 2x \\ 2y \\ 1-r^2 \end{pmatrix}$, $r^2 = x^2+y^2$. Let ϕ and θ be the usual spherical coordinates measuring the angle down from the positive z-axis, and horizontally



from the positive x-axis, i.e.

$$\phi = \cos^{-1}(z)$$

$$\theta = \tan^{-1}(y/x)$$

Then we can define vector fields $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ which measure the rate of change in the directions of the gradients of θ and ϕ respectively. In local coordinates using the chart h, we find

$$\frac{\partial}{\partial \theta} = \frac{1}{r^2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \phi} = \frac{2}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

Thus the vector fields $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ are defined in $S^2 - \{N, S\}$ where N and S are the "poles" $N = \begin{pmatrix}0\\0\\1\end{pmatrix}$ and $S = \begin{pmatrix}0\\0\\-1\end{pmatrix}$.

Definition: Vector fields $X_1, \ldots, X_k$ are linearly independent at m if the vectors

$$X_1(m), \ldots, X_k(m) \in T_m M$$

are linearly independent. They are linearly independent (or for emphasis, everywhere linearly independent) if they are linearly independent at each $m \in M$.

M is parallelizable if it has $m = \dim M$ linearly independent vector fields.

<u>Proposition</u>: M is parallelizable iff $\exists$ a diffeomorphism $\phi : TM \to M \times \mathbb{R}^m$ such that $\phi | T_xM$ is linear for each $x \in M$, and $\pi_1 \phi = \tau$.

$$\begin{tikzcd} TM \arrow[rr, "\phi"] \arrow[dr, "\tau"'] & & M \times \mathbb{R}^m \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

<u>Proof</u>: Given ϕ, define $X_i = \phi^{-1} \circ v_i$, $v_i(x) = (x, e_i)$.

Given $X_1, \dots, X_m$, any $v \in T_x M$ has a unique expression

$$v = \sum_{i=1}^{m} f_i X_i(x)$$

and we let $\phi(v) = (x, f_1, \dots, f_m)$. Using charts it is easy to see that ϕ is smooth if and only if each X_i is smooth. //

<u>Examples</u>:

① S^1 is parallelizable : $TS^1 \cong S^1 \times \mathbb{R}$. We need an everywhere nonzero vector field on S^1. The vector field

$$X\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_0 \end{pmatrix}, \quad x_0^2 + x_1^2 = 1, \text{ so } \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in S^1$$

will do.

② If we regard S^1 as $\{ z \in \mathbb{C} \mid |z| = 1 \}$ then $X(z) = iz$.

③ Regard S^{2n-1} as the unit sphere in $\mathbb{C}^n$. Then S^{2n-1} has an everywhere nonzero vector field

$$X(z_1, \dots, z_n) = (iz_1, \dots, iz_n),$$ or in real coordinates,

$$X(x_1, y_1, x_2, y_2, \dots, x_n, y_n) = (-y_1, x_1, -y_2, x_2, \dots, -y_n, x_n).$$

④ On S^5, the vector field

$$Y(x_1, y_1, x_2, y_2, x_3, y_3) = (-y_2, x_3, -y_3, x_1, -y_1, x_2)$$

is also a tangent vector field on S^5, but it is linearly dependent upon X at some points: $X = Y$ if $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$. It can be shown that <u>any</u> two vector fields on S^5 will be linearly dependent somewhere. See Example (7).

(5) S^3 is parallelizable. A set of three everywhere independent tangent vector fields is given by

$$X\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_0 \\ -x_3 \\ x_2 \end{pmatrix}, \quad Y\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_3 \\ x_0 \\ -x_1 \end{pmatrix}, \quad Z\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ x_1 \\ x_0 \end{pmatrix}$$

<u>Note</u>: Since $T_x S^3$ is three dimensional, it is sufficient to check that $X(x)$, $Y(x)$, and $Z(x)$ are linearly independent to see that they span.

It is simplest to consider S^3 as the quaternions of unit length. Then

$$X(q) = iq, \quad Y(q) = jq, \quad \text{and} \quad Z(q) = kq$$

are linearly independent because $\alpha X(q) + \beta Y(q) + \gamma Z(q) = 0$ implies $(\alpha i + \beta j + \gamma k)q = 0$, and since the quaternions are a division algebra and $q \neq 0$, it follows that $\alpha = \beta = \gamma = 0$.

(6) Consider S^{4n-1} as the unit sphere in $\mathbb{H}^n$, $\mathbb{H}$ = the quaternions. Then we have 3 tangent vector fields which are everywhere linearly independent:

$$X\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = i\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad Y\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = j\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad Z\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = k\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}.$$

(7) We will show shortly that S^{2n} has <u>no</u> everywhere nonzero vector fields. Using methods of algebraic topology, Frank Adams showed (~1961) that S^n has

exactly $\rho(n+1)-1$ linearly independent vector fields (Hurwitz and Radon had constructed this many; Adams showed there could be no more). Here

$$\rho(2^{4a+b} * \text{odd}) = 8a + 2^b \quad , \quad 0 \le b < 3.$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...
# l.i.v.f. on S^n	1	0	3	0	1	0	7	0	1	0	3	0	1	0	8	0	1	...

Cor: The only division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathcal{C}$, the Cayley numbers.

Proof: As for S^1 and S^3, a division algebra of dimension n would provide $n-1$ linearly independent vector fields on S^{n-1}. This occurs for $n = 1, 2, 4$ or 8 using $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathcal{C}$, but the above formula shows there are no other solutions.

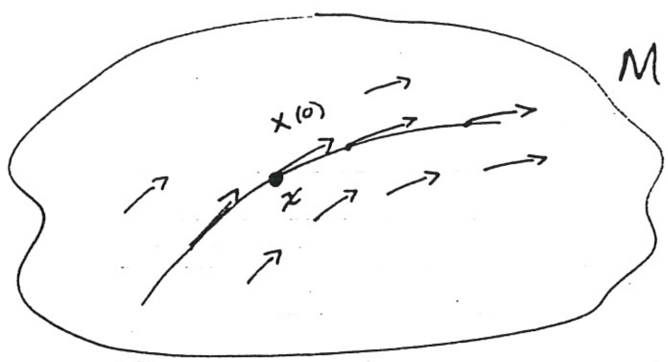
Remark: Lie groups will provide a large store of parallelizable manifolds (see the next section). Once we have shown this, it will follow that, unlike S^1 and S^3, which are groups with respect to multiplication of complex numbers and quaternions, the even spheres S^{2n} cannot support a group structure.

Application : Solving 1^{st} order PDEs by the method of characteristics

First, we need a brief account of the flow associated to a vector field $X : M \to TM$. Given $x \in M$, the integral curve of X through x is a map $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = x$, which is tangent to X at each point:

$$\gamma'(t) = X(\gamma(t))$$

or, more precisely, $\quad T\gamma(t,1) = X(\gamma(t))$



The flow associated to X is the collection of functions $F_t : M \to M$,

$$F_t(x) = \gamma(t)$$

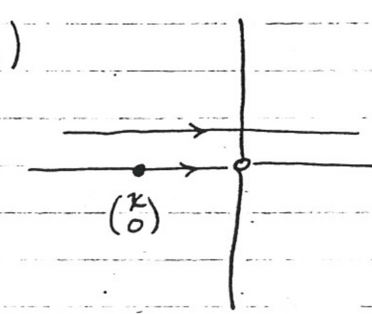
where γ is the integral curve to X through x. By the existence and uniqueness of solutions to 1^{st} order ODEs, F_t is defined for small t in any neighborhood, and by the smooth dependence of solutions on the initial conditions, each F_t is smooth. Further, we have the "semi-group" property

$$F_t \circ F_s = F_{t+s}$$

where these are defined.

Bad example : Let $M = \mathbb{R}^2 - \{\binom{0}{0}\}$ and $X\binom{x}{y} = \binom{1}{0}$

Then $F_t\binom{x}{y} = \binom{x+t}{y}$ except when $y = 0$ and $x < 0 < x+t$. The flow must stop at $\binom{0}{0}$ so $F_t\binom{x}{0}$ is defined only for $x + t < 0$, when $x < 0$.



Now, consider the first order linear PDE for $f: \mathbb{R}^n \to \mathbb{R}$:

$$\sum_{i=1}^{n} a_i(x,f)\frac{\partial f}{\partial x_i} = b(x,f) \qquad (E)$$

We have an associated vector field on $\mathbb{R}^{n+1}$

$$\sum_{i=1}^{n} a_i(x,f)\frac{\partial}{\partial x_i} + b(x,f)\frac{\partial}{\partial f} \qquad (V)$$

where we let f be our $(n+1)^{st}$ coordinate. Our initial conditions are specified by an $(n-1)$ dimensional submanifold $\Gamma \subset \mathbb{R}^{n+1}$ which is nowhere tangent to (V). If F_t is the flow associated to (V) then $F_t(\Gamma)$ defines f implicitly at time t; that is, it is the graph of f, solving (E).

<u>Example</u>:

$$2u_x + u_y + 3u = e^x y \qquad (E)$$

$$u(0,y) = \cos(y) \qquad (\Gamma)$$

The associated vector field is $\begin{pmatrix} 2 \\ 1 \\ e^x y - 3u \end{pmatrix} \quad (V)$

i.e.

$$x' = 2$$
$$y' = 1$$
$$u' = e^x y - 3u$$

This system of ODEs is easily solved:

$$x = x_0 + 2t$$
$$y = y_0 + t$$
$$u = \tfrac{1}{5}e^{x_0}(y_0 + t - \tfrac{1}{5})e^{2t} + Ce^{-3t}$$

The initial conditions Γ can be parameterized by $\begin{pmatrix} 0 \\ s \\ \cos(s) \end{pmatrix}$
Putting this into the solutions at $t=0$ yields

$x_0 = 0$, $y_0 = s$, $C = \cos(s) + \frac{1}{25} - \frac{1}{5}s$, which gives

$$x = 2t$$
$$y = s + t$$
$$u = \tfrac{1}{5}(s + t - \tfrac{1}{5})e^{2t} + (\cos(s) + \tfrac{1}{25} - \tfrac{1}{5}s)e^{-3t}$$

This gives the graph of u parametrically. We can solve for s and t,

$$t = x/2$$
$$s = y - x/2$$

and substitute to get the solution in explicit form:

$$u(x,y,t) = \frac{1}{5}\left(y - \frac{1}{5}\right)e^{x} + \left(\cos\left(y - \frac{x}{2}\right) - \frac{1}{5}\left(y - \frac{x}{2} - \frac{1}{5}\right)\right)e^{-3x/2}$$

(Reference: Abraham, Marsden, Ratiu, <u>Manifolds, Tensor Analysis, and Applications</u>)