Whitney's Theorem and related results

Our first application of Sard's Theorem is a proof of the easy whitney embedding theorem. We start by showing that any compact manifold embeds into Euclidean space. For this we will use a Lemma.

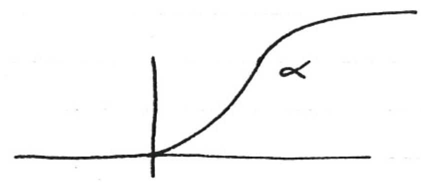
Lemma: For any positive real numbers $a < b$ there is a C^∞ function $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\lambda(x) = 1 \quad \text{if} \quad |x| \le a$$

$$0 < \lambda(x) < 1 \quad \text{if} \quad a < |x| < b$$

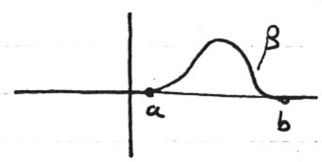
and

$$\lambda(x) = 0 \quad \text{if} \quad |x| \ge b.$$

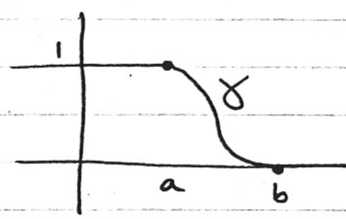


Proof: Let

$$\alpha(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$



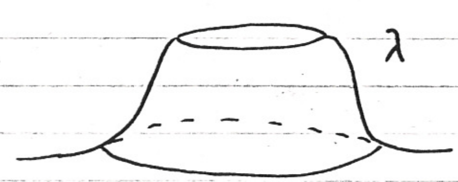
and $\beta(x) = \alpha(x-a)\,\alpha(b-x)$. It is simple to check that α and β are C^∞. Let



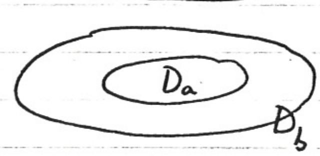
$$\gamma(x) = \frac{\int_x^b \beta}{\int_a^b \beta}$$

Finally,

$$\lambda(x) = \gamma(|x|) \,. //$$



We will refer to the functions λ as "bump functions", for obvious reasons.



Theorem: If M is a compact manifold, there is an embedding $M \longrightarrow \mathbb{R}^N$ for some N.

Proof: For each $x \in M$ pick a chart $h_x : D_3(0) \to M$, $h_x(0) = x$. Then $\{h_x(D_1(0))\}$ is an open cover of M, so there are points $x_1, \dots, x_d$ such that

$$\{h_{x_i}(D_1(0))\}_{i=1}^{d}$$

is an open cover of M. Let $\lambda : \mathbb{R}^m \to \mathbb{R}$, $m = \dim M$, be the bump function which is 1 inside $D_1(0)$ and 0 outside $D_2(0)$. Let

$$f_i : M \to \mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$$

be

$$f_i(x) = \begin{cases} (\lambda(h_{x_i}^{-1}(x)), \lambda(h_{x_i}^{-1}(x))\, h_{x_i}^{-1}(x)) & x \in \operatorname{Im} h_{x_i} \\ 0 & x \notin \operatorname{Im} h_{x_i}. \end{cases}$$

Then f_i is smooth on $h_{x_i}(D_3(0))$ since λ and $h_{x_i}^{-1}$ are smooth. Also, f_i is smooth on the complement of $h_{x_i}(\overline{D_2(0)})$ since it is constantly 0 there. Therefore f_i is smooth. Further $f_i = h_{x_i}^{-1}$, a diffeomorphism to its image, on $h_{x_i}(D_1(0))$.

Therefore Tf_i is 1-1 on $h_{x_i}(D_1(0))$. Now let

$$f : M \to \mathbb{R}^{d(m+1)}$$

be $f(x) = (f_1(x), \dots, f_d(x))$. This is an immersion since f_i is an immersion on $h_{x_i}(D_1(0))$ and $\{h_{x_i}(D_1(0))\}_{i=1}^{d}$ is a cover of M.

Finally, suppose $f(x) = f(y)$. Choose i so that $x \in h_{x_i}(D_1(0))$. Then $\lambda(h_{x_i}^{-1}(x)) = 1$ and since $f_i(x) = f_i(y)$ it follows that $\lambda(h_{x_i}^{-1}(y)) = 1$ also, so that $y \in \operatorname{Im} h_{x_i}$ as well. Therefore $h_{x_i}^{-1}(x) = h_{x_i}^{-1}(y)$ and hence $x = y$. //

Note: With a bit more work this could be extended to show that any manifold, compact or not, embeds in Euclidean space.

Next, we will show how to use Sard's Theorem to compress an embedding into a lower dimensional subspace.

<u>Theorem</u>: Given an injective immersion $f: M \longrightarrow \mathbb{R}^N$, $N > 1 + 2\dim(M)$, there is a linear projection $\pi: \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ such that $\pi f: M \longrightarrow \mathbb{R}^{N-1}$ is an injective immersion.

<u>Proof</u>: Let

$$h: M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^N \quad \text{by} \quad h(x,y,t) = t\,(f(x) - f(y))$$

and

$$g: TM \longrightarrow \mathbb{R}^N \quad \text{by} \quad g = \pi_2 \circ Df, \text{ that is,}$$

$$g(x,v) = Df(x)(v).$$

Since $N > 2m+1$, $m = \dim(M)$, a regular value of h or g is a point not in their image. By Sard's Theorem there exists $a \in \mathbb{R}^N$ which is a regular value for both h and g.

Let π be orthogonal projection onto the copy of $\mathbb{R}^{N-1} \subset \mathbb{R}^N$ which is orthogonal to a. Then $\pi(x) = \pi(y) \Rightarrow x - y$ is a multiple of a.

Now πf is 1-1 since $\pi f(x) = \pi f(y)$ implies $f(x) - f(y) = ta$ for some t, and if t were not 0, then $h(x,y,1/t) = a$, contradicting our choice of a. Thus $f(x) = f(y)$, and since f is injective, $x = y$.

Similarly πf is an immersion since the assumption that

$$\begin{aligned} 0 = D(\pi f)(x)(v) &= D\pi(f(x))(Df(x)(v)) \\ &= \pi\, Df(x)(v) \qquad \text{(since } \pi \text{ is linear)} \\ &= \pi\, g(x,v) \end{aligned}$$

implies $g(x,v) = ta$, and if $t \neq 0$ then $g(x, \frac{1}{t}v) = a$, again contradicting our choice of a. Thus $g(x,v) = 0$, but since f is an immersion, this implies $v = 0$. //

Note: This proof, using only g, not h, shows that there is an immersion $M \to \mathbb{R}^{2m}$.

Corollary: A compact m-dimensional manifold embeds in $\mathbb{R}^{2m+1}$ and immerses in $\mathbb{R}^{2m}$.

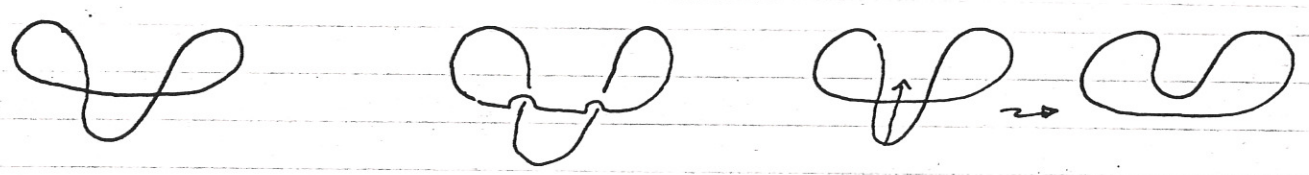
Note: With considerably more work Whitney showed

Whitney Embedding Theorem: Any m-dimensional manifold can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$.

To get this from what we have shown, two more steps are needed:

(1) remove the assumption of compactness from the first theorem (easy),
(2) push the embedding one dimension lower in the second theorem (hard).

To see why the second step might be hard, consider the immersion $S^1 \to \mathbb{R}^2$ shown below at the left. In $\mathbb{R}^3$ we can convert this to an embedding without moving anything more than ε, for any



chosen $\varepsilon > 0$, as indicated in the middle. However, to convert it to an embedding in $\mathbb{R}^2$ requires a large change, as shown on the right.