Here is an easy proof that $\mathbf{C}$ is algebraically closed.

Proof. Suppose E is a division algebra over $\mathbf{C}$ of degree $n < \infty$. Since E is a division algebra, multiplication gives a map

$$(E \setminus 0) \times (E \setminus 0) \to (E \setminus 0).$$

Since $\mathbf{C}$ is central in E and multiplication in E is $\mathbf{C}$-linear, it factors to give a map

$$\mu : \mathbf{CP}^{n-1} \times \mathbf{CP}^{n-1} \to \mathbf{CP}^{n-1}.$$

Claim 1. *$\mu^*(y) = 1 \otimes y + y \otimes 1$, where $y \in H^2\mathbf{CP}^{n-1}$ is a generator.*

Given this, we have

$$\begin{aligned} 0 &= \mu^*(y^n) \\ &= (1 \otimes y + y \otimes y)^n \\ &= 1 \otimes y^n + ny \otimes y^{n-1} + \cdots + y^n \otimes 1 \\ &= ny \otimes y^{n-1} + \cdots + ny^{n-1} \otimes y. \end{aligned}$$

from which it follows that $n = 1$. ◻

Proof of the claim. choose an $e \in E \setminus (\mathbf{C} \cdot 1)$. Then, restricting the multiplication, we have

$$(\mathbf{C} \cdot 1 \setminus 0) \times (\mathbf{C} \cdot \{1, e\} \setminus 0) \to (\mathbf{C} \cdot \{1, e\} \setminus 0)$$

inducing

$$\begin{array}{ccc} \mathbf{CP}^0 \times \mathbf{CP}^1 & \longrightarrow & \mathbf{CP}^1 \\ \downarrow & & \downarrow \\ \mathbf{CP}^{n-1} \times \mathbf{CP}^{n-1} & \longrightarrow & \mathbf{CP}^{n-1} \end{array}$$

The top map is a homeomorphism, in fact, essentially the identity, and the vertical maps are well known, showing that $1 \otimes y$ occurs with coefficient 1 in $\mu^*(y)$. The other term is handled similarly. ◻

Remark 2. *This is the integral version of the argument, commonly given in introductory algebraic topology courses, that a real division algebra must have dimension a power of two. Oddly, I have not found this in the literature.*