Here is an easy proof that \( \mathbb{C} \) is algebraically closed.

**Proof.** Suppose \( E \) is a division algebra over \( \mathbb{C} \) of degree \( n < \infty \). Since \( E \) is a division algebra, multiplication gives a map

\[
(E \setminus 0) \times (E \setminus 0) \to (E \setminus 0).
\]

Since \( \mathbb{C} \) is central in \( E \) and multiplication in \( E \) is \( \mathbb{C} \)-linear, it factors to give a map

\[
\mu : \mathbb{C} \mathbb{P}^{n-1} \times \mathbb{C} \mathbb{P}^{n-1} \to \mathbb{C} \mathbb{P}^{n-1}.
\]

**Claim 1.** \( \mu^*(y) = 1 \otimes y + y \otimes 1 \), where \( y \in H^2\mathbb{C} \mathbb{P}^{n-1} \) is a generator.

Given this, we have

\[
0 = \mu^*(y^n) = (1 \otimes y + y \otimes y)^n = 1 \otimes y^n + ny \otimes y^{n-1} + \cdots + y^n \otimes 1 = ny \otimes y^{n-1} + \cdots + ny^{n-1} \otimes y.
\]

from which it follows that \( n = 1 \). \( \square \)

**Proof of the claim.** choose an \( e \in E \setminus (\mathbb{C} \cdot 1) \). Then, restricting the multiplication, we have

\[
(\mathbb{C} \cdot 1 \setminus 0) \times (\mathbb{C} \cdot \{1, e\} \setminus 0) \to (\mathbb{C} \cdot \{1, e\} \setminus 0)
\]

inducing

\[
\begin{array}{ccc}
\mathbb{C} \mathbb{P}^0 \times \mathbb{C} \mathbb{P}^1 & \longrightarrow & \mathbb{C} \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathbb{C} \mathbb{P}^{n-1} \times \mathbb{C} \mathbb{P}^{n-1} & \longrightarrow & \mathbb{C} \mathbb{P}^{n-1}
\end{array}
\]

The top map is a homeomorphism, in fact, essentially the identity, and the vertical maps are well known, showing that \( 1 \otimes y \) occurs with coefficient 1 in \( \mu^*(y) \). The other term is handled similarly. \( \square \)

**Remark 2.** This is the integral version of the argument, commonly given in introductory algebraic topology courses, that a real division algebra must have dimension a power of two. Oddly, I have not found this in the literature.