THE COHOMOLOGY OF $ku$

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Recall that $H^* HZ = A/ASq^1$. We wish to show that $H^* ku = A/(Sq^1, Q_1)$. I don’t have my copy of Adams’ blue book handy, but this is what he does.

1. The $k$-invariant

We start with the (co)fiber sequence of spectra

$$
\Sigma^2 ku \xrightarrow{v} ku \xrightarrow{\epsilon} HZ
$$

which follows from the ring structure of $ku$ and $\pi_* ku = \mathbb{Z}[v]$. Iterating, we get the Postnikov tower of $ku$, with first $k$-invariant the top row in the diagram

\[
\begin{array}{ccc}
\Sigma^{-1} HZ & \xrightarrow{\Sigma^2 ku} & \Sigma^2 HZ \\
\downarrow{v} & & \downarrow{\epsilon} \\
k u & \longrightarrow & HZ
\end{array}
\]

We have to determine this $k$-invariant in $H^2 \Sigma^{-1} HZ = (Q_1)$

To do this, recall that $ku = (BU \times \mathbb{Z}, U, BU, SU, BSU, SU \langle 5 \rangle, BU \langle 6 \rangle, \ldots)$. Restricting to the 3rd spaces, $\Omega^\infty \Sigma^3(X)$, we have the diagram

\[
\begin{array}{ccc}
K(\mathbb{Z}, 2) & \xrightarrow{SU \langle 5 \rangle} & K(\mathbb{Z}, 5) \\
\downarrow{v} & & \downarrow{\epsilon} \\
SU & \longrightarrow & K(\mathbb{Z}, 3)
\end{array}
\]

The top row here is no help because $H^5 K(\mathbb{Z}, 2) = 0$, but we can detect more by looking instead at the boundary map for the fibration $SU \langle 5 \rangle \longrightarrow SU \longrightarrow K(\mathbb{Z}, 3)$:

$$
H^6 K(\mathbb{Z}, 3) \longleftrightarrow H^5 SU \langle 5 \rangle \leftrightarrow H^5 K(\mathbb{Z}, 5)
$$

If we let $x_5 \in H^5 SU \langle 5 \rangle$ and $\iota_3 \in H^3 K(\mathbb{Z}, 3)$ be the fundamental classes, then in total degree 5, the Serre spectral sequence for this fibration has two generators, $Sq^2 \iota_3$ and $x_5$. The only way that $H^5 SU$ can be one dimensional is for $d_5(x_5) = \iota_5^2$, giving the $k$-invariant:

$$
Q_1(\iota_3) = \iota_3^2 \leftrightarrow x_5 \leftrightarrow \iota_5
$$
Figure 1. Low degree classes in the $E_2$ term of the Serre spectral sequence $H^* K(\mathbb{Z}, 3) \otimes H^* SU(5) \Rightarrow H^* SU$.

Alternatively, we could look at the fourth spaces, $K(\mathbb{Z}, 3) \rightarrow BSU(6) \rightarrow BSU$, and compute

$H^6 K(\mathbb{Z}, 3) \leftarrow H^6 BSU(6) \leftarrow H^6 K(\mathbb{Z}, 6)$

In this fibration, we must have $d_3(\iota_3) = c_2$, and hence $d_5(Sq^2 \iota_3) = Sq^2 c_2 = c_3$. This leaves $\iota_3^2 \in H^6 K(\mathbb{Z}, 3)$ as the image of the generator of $H^6 BSU(6)$, giving the $k$-invariant:

$Q_1(\iota_3) = \iota_3^2 \leftarrow x_6 \leftarrow \iota_6$
THE COHOMOLOGY OF ku

\begin{figure}
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {$c_2$};
  \node (3) at (2,0) {$c_3$};
  \node (4) at (3,0) {$c_2^2$};
  \node (5) at (0,-1) {$\ell_3$};
  \node (6) at (1,-1) {$\ell_3^2c_2$};
  \node (7) at (2,-1) {$\ell_3^2c_3$};
  \node (8) at (3,-1) {$\ell_3^2c_2^2$};
  \node (9) at (0,-2) {$\text{Sq}^2\ell_3$};
  \node (10) at (1,-2) {$\text{Sq}^2\ell_3c_2$};
  \node (11) at (2,-2) {$\text{Sq}^2\ell_3c_3$};
  \node (12) at (3,-2) {$\text{Sq}^2\ell_3c_2^2$};
  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (3) to (4);
  \draw[->] (1) to (5);
  \draw[->] (5) to (6);
  \draw[->] (6) to (7);
  \draw[->] (7) to (8);
  \draw[->] (1) to (9);
  \draw[->] (9) to (10);
  \draw[->] (10) to (11);
  \draw[->] (11) to (12);
\end{tikzpicture}
\caption{Low degree classes in the $E_2$ term of the Serre spectral sequence $H^*BSU \otimes H^*K(\mathbb{Z}, 3) \Rightarrow H^*BSU(6)$. (Not all classes are shown.)}
\end{figure}

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\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {$c_2$};
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  \node (4) at (3,0) {$c_2^2$};
  \node (5) at (0,-1) {$\ell_3$};
  \node (6) at (1,-1) {$\ell_3c_2$};
  \node (7) at (2,-1) {$\ell_3c_3$};
  \node (8) at (3,-1) {$\ell_3c_2^2$};
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  \node (10) at (1,-2) {$\text{Sq}^2\ell_3c_2$};
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  \node (12) at (3,-2) {$\text{Sq}^2\ell_3c_2^2$};
  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (3) to (4);
  \draw[->] (1) to (5);
  \draw[->] (5) to (6);
  \draw[->] (6) to (7);
  \draw[->] (7) to (8);
  \draw[->] (1) to (9);
  \draw[->] (9) to (10);
  \draw[->] (10) to (11);
  \draw[->] (11) to (12);
\end{tikzpicture}
\caption{Low degree classes in the $E_4$ term of the Serre spectral sequence $H^*BSU \otimes H^*K(\mathbb{Z}, 3) \Rightarrow H^*BSU(6)$.}
\end{figure}
2. The Long Exact Sequence in Cohomology

From the k-invariant \( Q_1 \in H^3HZ \) for ku we get a diagram

\[
\begin{array}{cccccc}
\Sigma^{-1}HZ & \longrightarrow & \Sigma^2ku & \longrightarrow & ku & \longrightarrow & HZ & \longrightarrow & \Sigma^3ku \\
\Sigma^{-1}HZ & \xrightarrow{Q_1} & \Sigma^2HZ & \longrightarrow & FQ_1 & \longrightarrow & HZ & \xrightarrow{Q_1} & \Sigma^3HZ \\
\end{array}
\]

Factoring the homomorphism \( H^*HZ \xrightarrow{Q_1} \Sigma^3H^*HZ \) we get

\[
\begin{array}{cccccc}
A/E(1) & \longrightarrow & \Sigma^3A/E(1) & \longrightarrow & A/E(0) & \xleftarrow{Q_1} & \Sigma^3A/E(0) \\
\xleftarrow{\Sigma^3p} & & \xleftarrow{\Sigma^6A/E(1)} & & \xleftarrow{i} & & \xleftarrow{i} \Sigma^3A/E(0) \\
\end{array}
\]

To see the exactness claims here it is simplest to induce up from \( E(1) \)-modules, where the epi-mono factorization of \( Q_1 : \Sigma^3E(1)/E(0) \longrightarrow E(1)/E(0) \) and the short exact sequence are easy.

This allows us to factor the restriction \( H^*ku \hookrightarrow H^*HZ \) as

\[
H^*ku \xleftarrow{c} A/E(1) \xrightarrow{p} A/E(0) = H^*HZ
\]

This makes the left square below commute. The right outer square commutes because it is induced by maps of spectra. Inserting the factorization of the restriction \( \Sigma^3H^*ku \hookrightarrow \Sigma^3H^*HZ \), the rest of the diagram commutes because \( \Sigma^3p \) is an epimorphism.

\[
\begin{array}{cccccc}
H^*ku & \xleftarrow{c} & A/E(0) & \xrightarrow{\Sigma^3H^*ku} & \Sigma^3A/E(1) \\
\xleftarrow{\Sigma^3c} & & \xleftarrow{i} & & \xleftarrow{i} \Sigma^3A/E(0) \\
A/E(1) & \xleftarrow{p} & A/E(0) & \xleftarrow{Q_1} & A/E(0) \\
\end{array}
\]

We have finally constructed the map of exact sequences which allows us to inductively show that

\[
A/E(1) \xrightarrow{c} H^*ku
\]

is an isomorphism.

\[
\begin{array}{cccccc}
H^*\Sigma^2ku & \xleftarrow{H^*ku} & H^*HZ & \xleftarrow{q} & H^*\Sigma^3ku & \xleftarrow{H^*\Sigma ku} \\
\xleftarrow{c} & & \xleftarrow{\Sigma^2c} & & \xleftarrow{\Sigma^6c} & & \xleftarrow{Q_1} \Sigma^3A/E(0) \\
A/E(1) & \xleftarrow{p} & A/E(0) & \xleftarrow{i} & \Sigma^3A/E(1) & \xleftarrow{\Sigma^3p} \\
\end{array}
\]
Suppose that $c$ is an isomorphism in degrees less than or equal to $n$. Then $\Sigma^3c$ is an isomorphism in degrees $\leq n + 3$ and hence $c$ is a mono in degrees $\leq n + 3$, as follows. For $i \leq n + 3$, suppose $x \in (A/E(1))^i$ has $c(x) = 0$. Then $x = p(x')$ for some $x' \in (A/E(0))^i$ so $c(p(x')) = 0$ and thus $x' = q(y)$. Since $i \leq n + 3$, $y = \Sigma^3c(z)$ for some $z \in (\Sigma^3A/E(1))^i$. Then $x = pi(z) = 0$.

Similarly, $c$ is an epi in degrees $\leq n + 2$: for $i \leq n + 2$, suppose $x \in H^i\text{ku}$. Then its image $x' \in H^i\Sigma^2\text{ku} = H^{i+1}\Sigma^3\text{ku}$ is $\Sigma^3c(y)$ for some $y \in (\Sigma^3A/E(1))^{i+1}$. But $i(y) = q(x') = 0$ by exactness of the top row. Since $i$ is mono, $y = 0$, so $x' = 0$. This shows $x$ comes from $H^\ast HZ$, so is in the image of $c$.

The induction starts, trivially, so $c$ is an isomorphism in all degrees.

References