THE COHOMOLOGY OF ku

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Recall that $H^*H\mathbb{Z} = A/ASq^1$. We wish to show that $H^*\mathrm{ku} = A/A(Sq^1, Q_1)$.
I don't have my copy of Adams' blue book handy, but this is what he does.

1. Preliminary on $H\mathbb{Z}$

We want to show that $H^*H\mathbb{Z} = A/ASq^1 = A//E(0)$. Apply the Eilenberg MacLane functor H to the short exact sequence

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2.$$

to get a (co)fiber sequence $H\mathbb{Z} \to H\mathbb{Z} \to H\mathbb{Z}/2$. The map $2 : H\mathbb{Z} \to H\mathbb{Z}$ is 0 in mod 2 cohomology, so we get a short exact sequence (with $H = H\mathbb{Z}/2$)

$$(*) \qquad 0 \to H^{i-1}H\mathbb{Z} \to H^iH \to H^iH\mathbb{Z} \to 0.$$

At $i = 0$ we just have $H^0H \to H^0H\mathbb{Z}$, so this is iso. Then at $i = 1$, we have

$$0 \to H^0H\mathbb{Z} \to H^1H \to H^1H\mathbb{Z} \to 0,$$

and since $H^0H\mathbb{Z} = \mathbb{Z}/2$ by the $i = 0$ case, we have $H^1H\mathbb{Z} = 0$ and $H^0H\mathbb{Z} \to H^1H$ sends 1 to Sq^1. Since these are maps of modules over the Steenrod alegbra, the composite

$$H^iH \to H^iH\mathbb{Z} \to H^{i+1}H$$

is $a \mapsto aSq^1$.

Now, by (*), $H^*H\mathbb{Z}$ is A/I where I is the image of $H^{i-1}H\mathbb{Z} \to H^iH$. Composing with the epi $H^{i-1}H \to H^{i-1}H\mathbb{Z}$, the image stays the same. But we just showed the image of this composite is ASq^1.

2. The k-invariant

We start with the (co)fiber sequence of spectra

$$\Sigma^2\mathrm{ku} \xrightarrow{v} \mathrm{ku} \xrightarrow{\epsilon} H\mathbb{Z}$$

which follows from the ring structure of ku and $\pi_*\mathrm{ku} = \mathbb{Z}[v]$. Iterating, we get the Postnikov tower of ku, with first k-invariant the top row in the diagram

$$\begin{tikzcd}
\Sigma^{-1}H\mathbb{Z} \arrow[r] \arrow[rr, bend left] & \Sigma^2\mathrm{ku} \arrow[r, "\Sigma^2\epsilon"] \arrow[d, "v"] & \Sigma^2 H\mathbb{Z} \\
& \mathrm{ku} \arrow[r, "\epsilon"] & H\mathbb{Z}
\end{tikzcd}$$

We have to determine this k-invariant in

$$H\mathbb{Z}^2(\Sigma^{-1}H\mathbb{Z}) = \langle Q_1 \rangle.$$

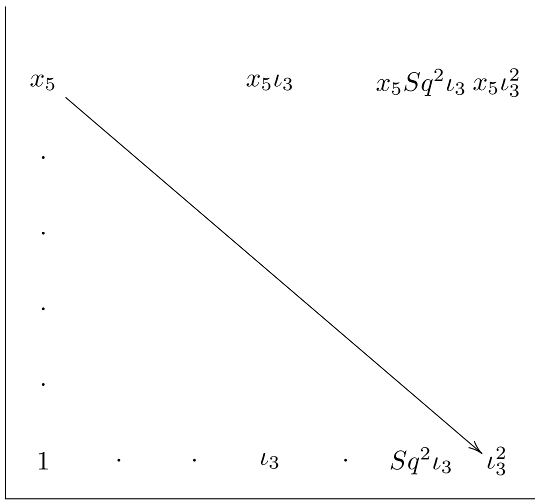

FIGURE 1. Low degree classes in the E_2 term of the Serre spectral sequence $H^*K(\mathbb{Z},3) \otimes H^*SU\langle 5\rangle \Longrightarrow H^*SU$.

To do this, recall that ku $= (BU \times \mathbb{Z}, U, BU, SU, BSU, SU\langle 5\rangle, BU\langle 6\rangle, \ldots)$. Restricting to the 3rd spaces, $\Omega^\infty\Sigma^3(X)$, we have the diagram

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[r] \arrow[rr, bend left] & SU\langle 5\rangle \arrow[r, "\Sigma^2\epsilon"] \arrow[d, "v"] & K(\mathbb{Z},5) \\ & SU \arrow[r, "\epsilon"] & K(\mathbb{Z},3) \end{tikzcd}$$

The top row here is no help because $H^5K(Z,2) = 0$, but we can detect more by looking instead at the boundary map for the fibration $SU\langle 5\rangle \longrightarrow SU \longrightarrow K(\mathbb{Z},3)$:

$$H^6K(\mathbb{Z},3) \longleftarrow H^5SU\langle 5\rangle \longleftarrow H^5K(\mathbb{Z},5)$$

If we let $x_5 \in H^5SU\langle 5\rangle$ and $\iota_3 \in H^3K(\mathbb{Z},3)$ be the fundamental classes, then in total degree 5, the Serre spectral sequence for this fibration has two generators, $Sq^2\iota_3$ and x_5. The only way that H^5SU can be one dimensional is for $d_5(x_5) = \iota_3^2$, giving the k-invariant:

$$Q_1(\iota_3) = \iota_3^2 \longleftarrow\!\shortmid x_5 \longleftarrow\!\shortmid \iota_5$$

Alternatively, we could look at the fourth spaces, $K(\mathbb{Z},3) \longrightarrow BSU\langle 6\rangle \longrightarrow BSU$, and compute

$$H^6K(\mathbb{Z},3) \longleftarrow H^6BSU\langle 6\rangle \longleftarrow H^6K(\mathbb{Z},6)$$

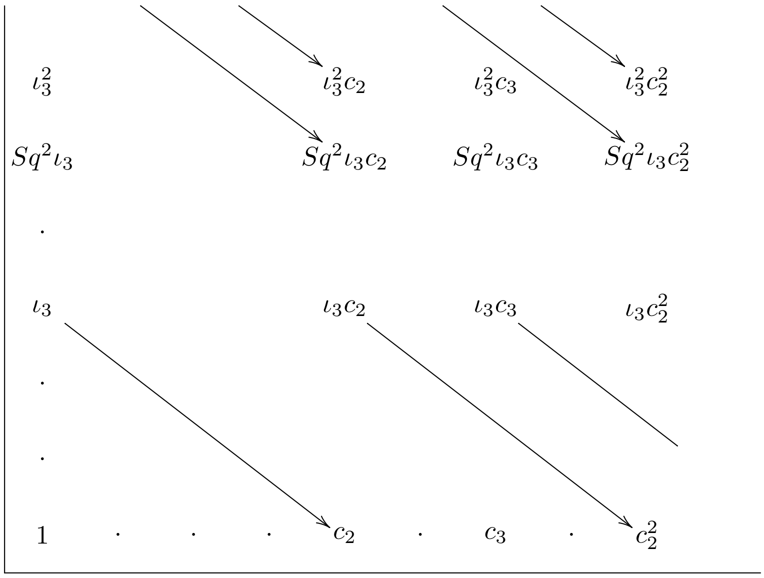

FIGURE 2. Low degree classes in the E_2 term of the Serre spectral sequence $H^*BSU \otimes H^*K(\mathbb{Z},3) \Longrightarrow H^*BSU\langle 6\rangle$. (Not all classes are shown.)

In this fibration, we must have $d_3(\iota_3) = c_2$, and hence $d_5(Sq^2\iota_3) = Sq^2c_2 = c_3$. This leaves $\iota_3^2 \in H^6K(\mathbb{Z},3)$ as the image of the generator of $H^6BSU\langle 6\rangle$, giving the k-invariant:

$$Q_1(\iota_3) = \iota_3^2 \longleftarrow\!\shortmid\; x_6 \longleftarrow\!\shortmid\; \iota_6$$

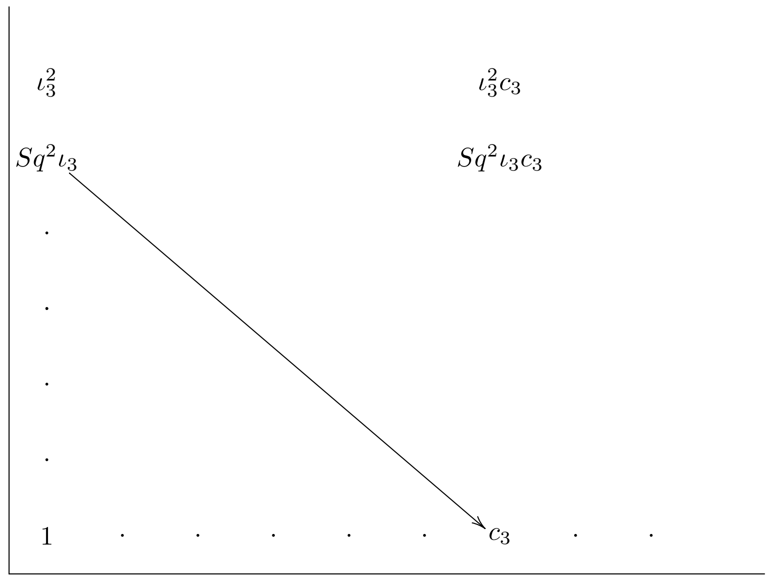

FIGURE 3. Low degree classes in the E_4 term of the Serre spectral sequence $H^*BSU \otimes H^*K(\mathbb{Z}, 3) \Longrightarrow H^*BSU\langle 6\rangle$.

3. The long exact sequence in cohomology

From the k-invariant $Q_1 \in H^3HZ$ for ku we get a diagram

$$
\begin{tikzcd}
\Sigma^{-1}H\mathbb{Z} \arrow[r] \arrow[d, equal] & \Sigma^2\mathrm{ku} \arrow[r] \arrow[d] & \mathrm{ku} \arrow[r] \arrow[d] & H\mathbb{Z} \arrow[r] \arrow[d, equal] & \Sigma^3\mathrm{ku} \arrow[d] \\
\Sigma^{-1}H\mathbb{Z} \arrow[r, "Q_1"] & \Sigma^2H\mathbb{Z} \arrow[r] & FQ_1 \arrow[r] & H\mathbb{Z} \arrow[r, "Q_1"] & \Sigma^3H\mathbb{Z}
\end{tikzcd}
$$

Factoring the homomorphism $H^*H\mathbb{Z} \xleftarrow{Q_1} \Sigma^3H^*H\mathbb{Z}$ we get

$$
\begin{tikzcd}
A/\!\!/E(1) & & \Sigma^3A/\!\!/E(1) \arrow[dl, hook', "i"'] & & \Sigma^6A/\!\!/E(1) \arrow[dl, hook', "\Sigma^3 i"'] \\
& A/\!\!/E(0) \arrow[ul, two heads, "p"] & & \Sigma^3A/\!\!/E(0) \arrow[ll, "Q_1"'] \arrow[ul, two heads, "\Sigma^3 p"'] &
\end{tikzcd}
$$

To see the exactness claims here it is simplest to induce up from $E(1)$-modules, where the epi-mono factorization of $Q_1 : \Sigma^3E(1)/\!\!/E(0) \longrightarrow E(1)/\!\!/E(0)$ and the short exact sequence are easy.

This allows us to factor the restriction $H^*\mathrm{ku} \longleftarrow H^*H\mathbb{Z}$ as

$$
H^*\mathrm{ku} \xleftarrow{\ c\ } A/\!\!/E(1) \overset{p}{\twoheadleftarrow} A/\!\!/E(0) = H^*H\mathbb{Z}
$$

This makes the left square below commute. The right outer square commutes because it is induced by maps of spectra. Inserting the factorization of the restriction $\Sigma^3H^*\mathrm{ku} \longleftarrow \Sigma^3H^*H\mathbb{Z}$, the rest of the diagram commutes because Σ^3p is an epimorphism.

$$
\begin{tikzcd}
H^*\mathrm{ku} & A/\!\!/E(0) \arrow[l] \arrow[dd, equal] & & \Sigma^3H^*\mathrm{ku} \arrow[ll] \\
& & \Sigma^3A/\!\!/E(1) \arrow[ur, "\Sigma^3 c"] \arrow[dl, hook', "i"'] & \\
A/\!\!/E(1) \arrow[uu, "c"] & A/\!\!/E(0) \arrow[l, two heads, "p"] & & A/\!\!/E(0) \arrow[ll, "Q_1"] \arrow[ul, two heads, "\Sigma^3 p"] \arrow[uu]
\end{tikzcd}
$$

We have finally constructed the map of exact sequences which allows us to inductively show that

$$
A/\!\!/E(1) \xrightarrow{c} H^*\mathrm{ku}
$$

is an isomorphism.

$$
\begin{tikzcd}
H^*\Sigma^2\mathrm{ku} & H^*\mathrm{ku} \arrow[l] & H^*HZ \arrow[l] \arrow[d, equal] & H^*\Sigma^3\mathrm{ku} \arrow[l, "q"'] & H^*\Sigma\mathrm{ku} \arrow[l] & {} \arrow[l] \\
& A/\!\!/E(1) \arrow[u, "c"] & A/\!\!/E(0) \arrow[l, two heads, "p"'] \arrow[d, equal] & \Sigma^3A/\!\!/E(1) \arrow[l, hook', "i"'] \arrow[u, "\Sigma^3 c"'] & & \\
& & A/\!\!/E(0) & \Sigma^3A/\!\!/E(0) \arrow[l, "Q_1"'] \arrow[u, two heads, "\Sigma^3 p"'] & &
\end{tikzcd}
$$

Suppose that c is an isomorphism in degrees less than or equal to n. Then $\Sigma^3 c$ is an isomorphism in degrees $\leq n+3$ and hence c is a mono in degrees $\leq n+3$, as follows. For $i \leq n+3$, suppose $x \in (A/\!\!/E(1))^i$ has $c(x) = 0$. Then $x = p(x')$ for some $x' \in (A/\!\!/E(0))^i$ so $c(p(x')) = 0$ and thus $x' = q(y)$. Since $i \leq n+3$, $y = \Sigma^3 c(z)$ for some $z \in (\Sigma^3 A/\!\!/E(1))^i$. Then $x = pi(z) = 0$.

Similarly, c is an epi in degrees $\leq n+2$: for $i \leq n+2$, suppose $x \in H^i\mathrm{ku}$. Then its image $x' \in H^i\Sigma^2\mathrm{ku} = H^{i+1}\Sigma^3\mathrm{ku}$ is $\Sigma^3 c(y)$ for some $y \in (\Sigma^3 A/\!\!/E(1))^{i+1}$. But $i(y) = q(x') = 0$ by exactness of the top row. Since i is mono, $y = 0$, so $x' = 0$. This shows x comes from H^*HZ, so is in the image of c.

The induction starts, trivially, so c is an isomorphism in all degrees, giving

$$\begin{tikzcd}
0 & H^*\mathrm{ku} \arrow[l] & H^*HZ \arrow[l] & H^*\Sigma^3\mathrm{ku} \arrow[l, "q"'] & 0 \arrow[l] \\
& A/\!\!/E(1) \arrow[u, "c", "\cong"'] & A/\!\!/E(0) \arrow[l, two heads, "p"] \arrow[u, equal] & \Sigma^3 A/\!\!/E(1) \arrow[l, tail, "i"] \arrow[u, "\Sigma^3 c", "\cong"'] &
\end{tikzcd}$$

References

[1] Frank Adams, Blue book, UofC Press.