The mod 2 Adams Spectral Sequence for $tmf_*$

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A report on joint work in progress with John Rognes.

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Introduction

- $tmf$-modules whose ordinary homology we can readily calculate, but whose $BP$ homology is much harder to get.
- The ordinary mod 2 Adams spectral sequence is thus a reasonable tool to understand these. We first needed to understand the ordinary Adams spectral sequence for $tmf$ itself in greater detail.
- Have $tmf$, $tmf \wedge C2$, $tmf \wedge C\eta$, and $tmf \wedge C\nu$ in gory detail.
- Tools: DMSS, Gröbner bases, ways to make calculations finite.
- Oddity: no Toda brackets needed except as a heuristic.
- Side benefit: independent verification of earlier results, with thorough documentation.
\[ \text{tmf}_* \]

Robert Bruner (WSU and UiO)
We may think of $ko$ as made up of two $H\mathbb{Z}$’s with a bit of 2-torsion tagging along.

In the same way, $tmf$ is made up of eight ‘$ko$’s’ with some $B$-torsion tagging along.

Here, $B \in \pi_8(tmf)$ maps to the Bott class in $\pi_8(ko)$ under a natural map $tmf \rightarrow ko$. We will call this lift to $tmf$ the ‘Bott class’ as well.
Generators of $tmf_*$

First, there is the periodicity element $M \in \pi_{192}(tmf)$, not a zero-divisor. Group remaining generators by $\sqrt{\text{Ann}_{\mathbb{Z}[B]}}$: for $0 \leq k \leq 7$,

(0) $D_k \in \pi_{24k}(tmf)$, $B_k \in \pi_{24k+8}(tmf)$, $C_k \in \pi_{24k+12}(tmf)$

(2) $\eta_k \in \pi_{24k+1}(tmf)$, and

(2, $B$) $\nu_k \in \pi_{24k+3}(tmf)$,
$\epsilon_k \in \pi_{24k+8}(tmf)$,
$\kappa_k \in \pi_{24k+14}(tmf)$, and
$\bar{\kappa}_k \in \pi_{24k+20}(tmf)$.

- $D_k$, $B_k$, $C_k$ and $\nu_k$ are defined for all $k$.
- The rest are defined only for some values of $k$. 
When $k = 0$, we have familiar elements:

- The unit $i : S \to \text{tmf}$ sends $1, \eta, \nu, \epsilon, \kappa$ and $\bar{\kappa}$ to $D_0 = 1, \eta_0, \nu_0, \epsilon_0, \kappa_0$ and $\bar{\kappa}_0$.

Omit the subscript 0 accordingly.

- The map $\text{tmf}_* \to \text{ko}_*$ sends $D_0 = 1, B_0 = B$, and $C_0 = C$ to generators of $\text{ko}_*$ in degrees 0, 8 and 12, respectively.

- The relation between $B$ and $\bar{\kappa}$ is the same as that between 2 and $\eta$, or between $\eta$ and $\nu$, a kind of `Frobenius’, detected by $Sq^0$ in Ext.

- $\kappa^2 = B\bar{\kappa}$. 
All the generators except $M$:

<table>
<thead>
<tr>
<th>Start+</th>
<th>0</th>
<th>8</th>
<th>12</th>
<th>1</th>
<th>3</th>
<th>8</th>
<th>14</th>
<th>20</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$B$</td>
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<td>$\nu$</td>
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<tr>
<td>24</td>
<td>$D_1$</td>
<td>$B_1$</td>
<td>$C_1$</td>
<td>$\eta_1$</td>
<td>$\nu_1$</td>
<td>$\epsilon_1$</td>
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<tr>
<td>48</td>
<td>$D_2$</td>
<td>$B_2$</td>
<td>$C_2$</td>
<td>$\nu_2$</td>
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<tr>
<td>72</td>
<td>$D_3$</td>
<td>$B_3$</td>
<td>$C_3$</td>
<td>$\eta_4$</td>
<td>$\nu_4$</td>
<td>$\epsilon_4$</td>
<td>$\kappa_4$</td>
<td>($\bar{\kappa}_4$)</td>
</tr>
<tr>
<td>96</td>
<td>$D_4$</td>
<td>$B_4$</td>
<td>$C_4$</td>
<td>$\nu_5$</td>
<td>$\epsilon_5$</td>
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</tr>
<tr>
<td>120</td>
<td>$D_5$</td>
<td>$B_5$</td>
<td>$C_5$</td>
<td>$\nu_6$</td>
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</tr>
<tr>
<td>144</td>
<td>$D_6$</td>
<td>$B_6$</td>
<td>$C_6$</td>
<td>($\nu_7$)</td>
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</tr>
<tr>
<td>168</td>
<td>$D_7$</td>
<td>$B_7$</td>
<td>$C_7$</td>
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Here, $\nu_3 = \eta_1^3$, $\bar{\kappa}_4 = \bar{\kappa}D_4$ and $\nu_7 = 0$ are not needed to generate $\pi_*(\text{tmf})$, but are convenient in general formulas.
Families of elements in $tmf_*$

- Consider the natural map $tmf_* \rightarrow MF_*/2$ to the ring of modular forms

$$MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(1728\Delta = c_4^3 - c_6^2) = \mathbb{Z}[c_4, \Delta][\sqrt{c_4^3 - 12^3\Delta}]$$

- The discriminant $\Delta$ is not in the image of the map to $MF_*/2$, but it does exist in the spectral sequences leading to $tmf_*$

- Differential on $\Delta$ kills $\nu \bar{K}$, detected by $h_2g$ in the Adams spectral sequence, giving Massey products at $E_2$ of the Adams spectral sequence

$$\Delta(x) = \langle h_2, g, x \rangle$$

$$\Delta'(x) = \langle g, h_2, x \rangle$$
Theorem

Repeated application of $\Delta$ gives classes detecting the following sequences of elements of $\pi_\ast(tmf)$:

- $8D \leftrightarrow D_1 \leftrightarrow 2D_2 \leftrightarrow D_3 \leftrightarrow 4D_4 \leftrightarrow D_5 \leftrightarrow 2D_6 \leftrightarrow D_7 \leftrightarrow 8M$.
- $C \leftrightarrow C_1 \leftrightarrow C_2 \leftrightarrow C_3 \leftrightarrow C_4 \leftrightarrow C_5 \leftrightarrow C_6 \leftrightarrow C_7 \leftrightarrow CM$.
- $B + \epsilon \leftrightarrow B_1 + \epsilon_1 \leftrightarrow B_2 \leftrightarrow B_3 \leftrightarrow B_4 + \epsilon_4 \leftrightarrow B_5 + \epsilon_5 \leftrightarrow B_6 \leftrightarrow B_7 \leftrightarrow BM$.
- $8B \leftrightarrow 8B_1 \leftrightarrow 8B_2 \leftrightarrow 8B_3 \leftrightarrow 8B_4 \leftrightarrow 8B_5 \leftrightarrow 8B_6 \leftrightarrow 8B_7 \leftrightarrow 8BM$. 
$\Delta'$ on $h_1$ and $h_2$ in $\text{Ext}$

\[
\begin{array}{ccccccc}
  h_1 & \to & h_1 & \to & h_0 & \leftarrow & h_0 \\
  h_1 & \downarrow & h_1 & \downarrow & h_0h_2 & \downarrow & h_2 \\
  h_1w_2 & \downarrow & \{h_1^2w_2, \gamma^2\} & \downarrow & h_1\gamma^2 = h_0^2h_2w_2 & \downarrow & h_0h_2w_2 \\
  h_1w_3^2 & \downarrow & \{h_1^2w_3^2, \gamma^2w_2\} & \downarrow & h_1\gamma^2w_2 = h_0^2h_2w_3^2 & \downarrow & h_0h_2w_3^2 \\
  h_1w_3^3 & \downarrow & \{h_1^2w_3^3, \gamma^2w_2\} & \downarrow & h_1\gamma^2w_3^2 = h_0^2h_2w_3^3 & \downarrow & h_0h_2w_3^3 \\
  h_1w_4 & \downarrow & \{h_1^2w_4, \gamma^2w_2\} & \downarrow & h_1\gamma^2w_3^3 = h_0^2h_2w_3^4 & \downarrow & h_0h_2w_3^4 \\
\end{array}
\]
Effect on $\eta_i$ and $\nu_i$

\[
\begin{array}{cccccc}
\eta & \eta^2 & \eta^3 = 4\nu & 2\nu & \nu \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\eta_1 & \eta_1 & \eta_2 = 2\nu_1 & \nu_1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\eta_1^2 & \eta_1^2 & \eta_2^2 = 4\nu_2 & 2\nu_2 & \nu_2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\eta_1^3 & \eta_1^3 & \\
\downarrow & \downarrow & \\
\eta_4 & \eta_4 & \eta_2^2 = 4\nu_4 & 2\nu_4 & \nu_4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\eta_1\eta_4 & \eta_1\eta_4 & \eta_1\eta_4 = 2\nu_5 & \nu_5 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\eta_1^2\eta_4 & \eta_1^2\eta_4 = 4\nu_6 & 2\nu_6 & \nu_6 \\
\end{array}
\]
Let $\Gamma_B(tmf_*)$ be the $B$-power torsion, those $x \in tmf_*$ such that $B^ix = 0$ for $i \gg 0$.

**Theorem**

$\nu_k, \epsilon_k, \kappa_k, \bar{\kappa} \in \Gamma_B(tmf_*)$.

**Theorem**

If $x \in \Gamma_B(tmf_*)$ then $Bx = \epsilon x$. 
General principle:

if \( x, y \in \{\eta, \nu, \epsilon, \kappa\} \) then \( x_i y_j \) depends only on \( x, y, \) and \( i + j \).

Exceptions stem from the varying 2-divisibility of the \( 24k + 3 \) stem. For example,

\[
\nu_1\nu_k = \begin{cases} 
2\nu \nu_{k+1} & k = 1, 5 \\
0 & \text{other } k \leq 7
\end{cases}
\]

\[
\nu_2\nu_2 = \nu \nu_4, \, \nu_2\nu_4 = \nu \nu_6, \text{ and } \nu_2\nu_6 = \nu \nu_8 = \nu^2 M.
\]
(In general, let \( x_{k+8} = x_k M \).)

\[
\nu_4\nu_6 = \pm \nu \nu_2 M.
\]
24 to 48
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48 to 72

\[ \text{Diagram showing points labeled } \eta_1, \eta_2, \kappa, \nu, \epsilon, \bar{\kappa}, \text{ etc.} \]
72 to 96
96 to 120
120 to 144
144 to 168
168 to 192
Definition

If \( x \in \pi_d R \) and \( M \) is an \( R \)-module, let

\[
M[1/x] = \text{hocolim} \left( M \xrightarrow{x} \Sigma^{-d} M \xrightarrow{x} \Sigma^{-2d} M \xrightarrow{x} \ldots \right)
\]

and let \( M/x^\infty \) be the homotopy cofiber of \( M \longrightarrow M[1/x] \).

We can iterate this to get \( M/(x^\infty, y^\infty) = M/x^\infty \wedge_R M/y^\infty \), etcetera.

We get

\[
0 \longrightarrow \pi_*(M)/x^\infty \longrightarrow \pi_*(M/x^\infty) \longrightarrow \Gamma_x \pi_{*-1}(M) \longrightarrow 0
\]

Theorem

\[
\Sigma^{20} \text{tmf} \simeq I(\text{tmf}/(2^\infty, B^\infty, M^\infty))
\]
Sketch proof:

- Let $N_\ast$ be the $\mathbb{Z}[B]$ submodule of $\pi_\ast(tmf)$ generated in degrees less than 192 (equivalently, $\leq 180$).
- $\Gamma_B N_\ast$ is zero outside $3 \leq \ast \leq 164$
- $N_\ast/B^\infty$ is zero above dimension 172 and is $\mathbb{Z}$, generated by $C_7/B$, in degree 172.
- Multiplication $N_\ast \otimes \mathbb{Z}[M] \rightarrow \pi_\ast(tmf)$ is a $\mathbb{Z}[B, M]$-isomorphism.
- $\Gamma_M \pi_\ast(tmf) = 0$
- $N_\ast \otimes \mathbb{Z}[M]/M^\infty \cong \pi_\ast(tmf)/M^\infty \cong \pi_\ast(tmf/M^\infty)$
- Short exact sequence

$$0 \rightarrow N_\ast/B^\infty \otimes \mathbb{Z}[M]/M^\infty \rightarrow \pi_\ast(tmf/(B^\infty, M^\infty))$$

$$\rightarrow \Gamma_B N_{\ast-1} \otimes \mathbb{Z}[M]/M^\infty \rightarrow 0$$
Sketch proof: (cont)

- \( \pi_*(tmf/(B^\infty, M^\infty)) \) is concentrated in degrees \( \leq -20 \) and in degree \(-20\) is \( \mathbb{Z} \) generated by \( C_7/BM \).
- It is 0 in degree -21, so the short exact sequence

\[
0 \longrightarrow \pi_*(tmf/(B^\infty, M^\infty))/2^\infty \longrightarrow \pi_*(tmf/(2^\infty, B^\infty, M^\infty)) \longrightarrow \Gamma_2\pi_*^{-1}(tmf/(B^\infty, M^\infty)) \longrightarrow 0
\]

implies that \( \pi_*(tmf/(2^\infty, B^\infty, M^\infty)) \) is concentrated in degrees \( \leq -20 \) and in degree \(-20\) is \( \mathbb{Z}/2^\infty \).
- \( \pi_*(I(tmf/(2^\infty, B^\infty, M^\infty))) \) is concentrated in degrees \( \geq 20 \) and is \( \mathbb{Z}_2 \) in degree 20.
- Choosing a 2-adic generator, we get a \( tmf \)-module map inducing an isomorphism in \( \pi_{20} \) between 20-connected spectra

\[
\Sigma^{20} tmf \longrightarrow I(tmf/(2^\infty, B^\infty, M^\infty))
\]

- It is an equivalence by inducing along \( tmf \longrightarrow BP\langle 2 \rangle \)
As usual, the equivalence of spectra yields isomorphisms (and pairings) with different shifts on the homotopy. Filter $\pi_\ast(tmf)$ by

$$
\Delta \pi_\ast(tmf) \subset \Gamma_B \pi_\ast(tmf) \subset \Gamma_2 \pi_\ast(tmf) \subset \pi_\ast(tmf)
$$

where

- $\Delta$ is the submodule of $\Gamma_B$ generated by the classes not in degrees 3 (mod 24), and
- $ko[k]$ is the $\mathbb{Z}[B]$ submodule generated by $\{D_k, B_k, C_k\}$ together with the appropriate $\eta$'s:

0: $\eta, \eta^2$

1: $\eta_1, \eta \eta_1$

2: $\eta B_2, \eta_1^2$

3: $\eta B_3, \eta^2 B_3$

4: $\eta_4, \eta \eta_4$

5: $\eta B_5, \eta_1 \eta_4$

6: $\eta B_6, \eta^2 B_6$

7: $\eta B_7, \eta^2 B_7$
Proposition

As a $\mathbb{Z}[B, M]$ module

$$\frac{\Gamma_B \pi_* (tmf)}{\Delta \pi_* (tmf)} \cong \bigoplus_{k=0}^{7} \langle \nu_k \rangle \otimes \mathbb{Z}[M]$$

and

$$\frac{\pi_* (tmf)}{\Gamma_B \pi_* (tmf)} \cong \bigoplus_{k=0}^{7} ko[k] \otimes \mathbb{Z}[M]$$
Duality in the $B$-torsion

Figure: Duality between $\Delta[0]$ and $\Delta[6]$
<table>
<thead>
<tr>
<th>ν₁</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>35</td>
</tr>
<tr>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

| ε₁ κ | 32 | 35 |
| ϵ₁ | 35 | 39 |
| κν₁ | 39 | 42 |
| η₁ | 42 |

| ν₅ | 142 | 138 | 135 |
| ε₅ κ | 142 | 138 | 135 |
| ϵ₅ | 138 | 135 | 131 |
| κν₅ | 135 | 131 | 128 |
| η₁ κ₄ | 131 | 128 | 125 |
| ε₅ | 128 | 125 | 123 |

**Figure:** Duality between Δ[1] and Δ[5]
Figure: Duality between \( \Delta[2] \) and \( \Delta[4] \)
Figure: Self-duality of $\Delta[3]$
Proposition

\[ \langle \nu_{7-k} \rangle \text{ is Pontrjagin 171-dual to } \langle B_k/B \rangle: \]

\[
\bigoplus_{k=0}^{7} \langle \nu_k \rangle_{171-*} \cong \text{Hom} \left( \bigoplus_{k=0}^{7} \langle B_k/B \rangle, \mathbb{Q}/\mathbb{Z} \right)
\]

(Note however that \( \nu_7 = 0 \) and \( 0 = \langle B_0/B \rangle \subset ko[0]/B^\infty \).)

Further, in \( \pi_*(tmf/B^\infty) \) the class which maps to \( \nu_k \in \Gamma_B \pi_*(tmf) \) lifts to a class \( \tilde{\nu}_k \) with \( 2^j \tilde{\nu}_k = C_k/B \).
Maps to $MF_{*/2}$

- The elliptic spectral sequence of Hopkins (2002) has edge hom
  \[ e: \pi_*(tmf) \longrightarrow MF_{*/2} = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 1728\Delta) \]

- $MF_{*/2}$ is the ring of integral modular forms, with $c_4$, $c_6$ and $\Delta$ in weights $*/2 = 4$, 6 and 12, corresponding to topological degrees $* = 8$, 12 and 24.

- By Hopkins (2002) and Bauer (2008), $\text{im}(e)$ is additively
  $\mathbb{Z}\{a_{i,j,k}c_4^i c_6^j \Delta^k | i \geq 0, j \in \{0, 1\}, k \geq 0\}$ where
  \[ a_{i,j,k} = \begin{cases} 24/\gcd(k, 24) & \text{for } i = 0 \text{ and } j = 0, \\ 1 & \text{for } i \geq 1 \text{ and } j = 0, \\ 2 & \text{for } j = 1. \end{cases} \]

  This is an integral result. See also Douglas-Francis-Henriques-Hill (2014) and Konter (2012).
Proposition

- \( \ker(e) = \Gamma_{2\pi_*}(tmf) \).
- \( B_k \) can be chosen to map to \( c_4 \Delta^k \), for each \( 0 \leq k \leq 7 \).
- \( C_k \) can be chosen to map to \( 2c_6 \Delta^k \), for each \( 0 \leq k \leq 7 \).
- \( D_k \) can be chosen to map to \( 2^i \Delta^k \) for \( 1 \leq k \leq 7 \), where

\[
i = \begin{cases} 
3 & k \equiv 1 \pmod{2} \\
2 & k \equiv 2 \pmod{4} \\
1 & k = 4 
\end{cases}
\]

(We complete at 2 here.)

- The image in \( MF_{*/2} \) and the Adams representative in \( E_{\infty}(tmf) \) uniquely determines each of \( B_k, C_k \) and \( D_k \) in \( \pi_*(tmf) \), with the exception of \( C_2, B_3 \) and \( C_6 \). In each case, the ambiguity is a class of order 2.
• $C_2$ is determined modulo $2\tilde{\kappa}^3 = \nu^3 \nu_2 = \eta \epsilon \nu_2$,

• $B_3$ is determined modulo $\tilde{\kappa}^4$, and

• $C_6$ is determined modulo $\nu^3 \nu_6 = \eta \epsilon \nu_6$. 
The Davis-Mahowald spectral sequence is a substitute for the Cartan-Eilenberg spectral sequence when the sub Hopf algebra is not normal.

In Davis and Mahowald (1982) the multiplicative structure is a somewhat \textit{ad hoc} afterthought. We give precise conditions for it.

- $\Gamma$, Hopf algebra over $k$
- $\Lambda \subset \Gamma$, sub Hopf algebra
- $\Omega := \Gamma \left/ \Lambda \right. = \Gamma \otimes_{\Lambda} k$ is a quotient $\Gamma$-module coalgebra
DMSS, dual formulation

- $\Gamma_*$ commutative Hopf algebra.
- $\Lambda_*$ quotient Hopf algebra of $\Gamma$.
- $\Omega_* = \Gamma_* \square_{\Lambda_*} k$ left $\Gamma_*$-comodule algebra.
- Require (suitable) $\Gamma_*$-comodule algebra resolution $k \to (\Omega_* \otimes R^*, d)$.
- Get multiplicative Davis–Mahowald spectral sequence

$$E_1^{\sigma,s,*} = \text{Ext}_{\Lambda_*}^{s,*}(k, R^\sigma) \Longrightarrow \sigma \text{Ext}_{\Gamma_*}^{s+\sigma,*}(k, k).$$

- Untwisting $\Omega_* \otimes R^\sigma \cong \Gamma_* \square_{\Lambda_*} R^\sigma$ is multiplicative for commutative $\Gamma_*$. 
DMSS, dual formulation, cont.

- Assume a graded $\Gamma^*$-comodule algebra $R^* = \bigoplus_{\sigma} R^\sigma$ and homomorphisms $d : \Omega^* \otimes R^\sigma \to \Omega^* \otimes R^{\sigma+1}$
- such that $(\Omega^* \otimes R^*, d)$ is a differential graded $\Gamma^*$-comodule algebra and the unit $k \to (\Omega^* \otimes R^*, d)$ is a quasi-isomorphism.
- Get an algebra spectral sequence

$$E_{1}^{\sigma,s} = \text{Ext}^{s}_{\Lambda^*}(k, R^\sigma) \Longrightarrow_{\sigma} \text{Ext}^{s+\sigma}_{\Gamma^*}(k, k)$$

- Product $E_{1}^{\sigma,*,*} \otimes E_{1}^{\tau,*,*} \to E_{1}^{\sigma+\tau,*,*}$ equals pairing induced by $\Lambda^*$-comodule product $R^\sigma \otimes R^\tau \to R^{\sigma+\tau}$. 
Main example: $A(2)$

- Commutative Hopf algebras

\[ \Gamma_* = A(2)_* \rightarrow A(1)_* = \Lambda_* \]

i.e.,

\[ \mathbb{F}_2[\xi_1, \bar{\xi}_2, \bar{\xi}_3]/(\xi_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2) \rightarrow \mathbb{F}_2[\xi_1, \bar{\xi}_2]/(\xi_1^4, \bar{\xi}_2^2) \]

- The left $A(2)_*$-comodule algebra

\[ \Omega_* = A(2)_* \square_{A(1)_*} \mathbb{F}_2 = E[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3] \]

is a sub $A(2)_*$-comodule algebra, but not a sub coalgebra.
Main example, cont.

- Resolve by $A(2)_*\text{-comodule algebra} \ R^* = \mathbb{F}_2[x_4, x_6, x_7]$ with coaction

$$\nu(x_4) = 1 \otimes x_4$$
$$\nu(x_6) = 1 \otimes x_6 + \xi_1^2 \otimes x_4$$
$$\nu(x_7) = 1 \otimes x_7 + \xi_1 \otimes x_6 + \bar{\xi}_2 \otimes x_4.$$

- Resolution $\mathbb{F}_2 \to \Omega_* \otimes R^*$ has differential

$$d(\xi_1^4) = x_4$$
$$d(\bar{\xi}_2^2) = x_6$$
$$d(\bar{\xi}_3) = x_7$$
The Davis-Mahowald spectral sequence

\[ \sigma = 0 \]

- \( R^0 = \mathbb{F}_2 \)
- \( E^{0,*,*}_1 = \text{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) = ko^{*,*} \).
\[ \sigma = 1 \]

- \[ R^1 = \mathbb{F}_2 \{ x_4, x_6, x_7 \} = \Sigma^4 H_*(S \cup \eta e^2 \cup 2 e^3) \].
- \[ E^{1,*,*}_1 = \text{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, R^1) = ksp^{*,*}\{h_2\} \].
The Davis-Mahowald spectral sequence

\[ \sigma = 2 \]

- \[ R^2 = \mathbb{F}_2 \{ x_4^2, x_4 x_6, x_4 x_7, x_6^2, x_6 x_7, x_7^2 \} \].
- \[ E_2^{2,*,*} = \text{Ext}_{A(1)_*}^*(\mathbb{F}_2, R^2) = G_2^{*,*} \{ h_2^2 \} \].
\[ \sigma = 3 \]

- \( \dim R^3 = 10 \).
- \( E_{1}^{3,**} = \text{Ext}_{A(1)}^{**,\ast}(\mathbb{F}_2, R^3) = G_{3,\ast}^{\ast} \{ h_2^3 \} \).
Theorem (Shimada and Iwai)

The cohomology of $A(2)$ is

$$\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, d_0, e_0, g, \alpha, \beta, \gamma, \delta, w_1, w_2]/I.$$ 

The ideal $I$ has 54 generators:

- $h_0h_1, h_1h_2, h_0^2h_2 - h_1^3, h_0h_2^2, h_2^3$
- $\ldots$
- $c_0\gamma - h_1\delta, \beta\gamma - g^2, d_0^2 - gw_1, \gamma\delta - h_1c_0w_2$,
- $\gamma^2 - h_1^2w_2 - g\beta^2, \alpha^4 - h_0^4w_2 - w_1g^2$
Free over $\mathbb{F}_2[w_1, w_2]$; here $w_1$ and $w_2$ restrict to $v_1^4$ and $v_2^8$, resp.

A sum of cyclic $R = \mathbb{F}_2[g, w_1, w_2]$-modules isomorphic to $R$, $R/(g)$ and $R/(g^2)$.

Four infinite families, $h_0^i \alpha^j$, $i \geq 0$, $0 \leq j \leq 3$.

Thirty-two other summands.

$E_3$, $E_4$ and $E_5 = E_\infty$ are then modules over $R_1 = \mathbb{F}_2[g, w_1, w_2^2]$ and $R_2 = \mathbb{F}_2[g, w_1, w_2^4]$ resp. Mostly cyclic.
$R_0$ generators of $\text{Ext}_A(2)(\mathbb{F}_2, \mathbb{F}_2)$

No circle indicates an $R_0$, one circle an $R_0/(g)$, and two circles an $R_0/(g^2)$.
First differentials

Squaring operations in Ext quickly give us quite a few differentials.

- $d_2(\alpha) = h_2 w_1$ and $d_2(\beta) = h_0 d_0$
- $d_3(\alpha^2) = h_1 d_0 w_1$ and $d_3(\beta^2) = h_1 g w_1$
- $d_3(w_2^2) = Sq^9(d_2(w_2))$

From these many others follow by the Leibniz rule.
Key differentials

There are three *hard* differentials, from which we can deduce everything else using the product structure. They are

**Theorem**

- $d_3(e_0) = c_0 w_1$
- $d_4(e_0 g) = gw_1^2$
- $d_2(w_2) = \alpha \beta g$
\[ d_3(e_0) = c_0 w_1 \]

- The Im(J) generator \( \rho \in \pi_{15}(S) \) in Adams filtration 4 must either map to 0 or \( \eta \kappa \) in \( \pi_{15}(tmf) \).
- \( \eta \rho \in \pi_*(S) \) is detected by \( \{ Pc_0 \} \) in \( \pi_*(S) \), which maps to \( c_0 w_1 \) in \( \text{Ext}_{A(2)}(F_2, F_2) \).
- \( \eta^2 \kappa = 0 \) in \( \pi_*(S) \) (Toda).
- \( c_0 w_1 \) must be a boundary and \( d_3(e_0) \) is the only chance.
0 to 24

Key Differentials

Robert Bruner (WSU and UiO)  
\(\text{tmf}_* \text{ at } p = 2\)  
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\[ d_4(e_0g) = gw_1^2 \]

- \( \eta^2 \bar{\kappa} \) is detected by \( Pd_0 \) in \( \pi_{22}(S) \) (Barratt-Mahowald-Tangora, Mimura?). This maps to \( d_0w_1 \) in \( \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \).
- \( \kappa \cdot \eta^2 \bar{\kappa} = 0 \) since \( \eta^2 \kappa = 0 \)
- \( \kappa \cdot \eta^2 \bar{\kappa} \) is detected by \( d_0 \cdot Pd_0 \) which maps to \( d_0^2w_1 = gw_1^2 \) in \( \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \).
- \( d_4(e_0g) \) is the only class which can hit it.
Most differentials not shown
$d_2(w_2) = \alpha \beta g$

**Corollary**

$d_4(d_0 e_0) = d_0 w_1^2$ and $d_4(\beta^2 g) = \alpha^2 e_0 w_1$ and these are both nonzero.

**Theorem**

$d_4(h_1^2 w_2) = \alpha^2 e_0 w_1$, $d_2(w_2) = \alpha \beta g$, and $d_3(h_1 w_2) = g^2 w_1$.

- $\gamma$ must live to at least $E_6$, so $d_4(\gamma^2) = 0$
- $\gamma^2 = h_1^2 w_2 + \beta^2 g$, so $d_4(h_1^2 w_2) = \alpha^2 e_0 w_1 \neq 0$
- If $d_2(w_2) = 0$ then $d_4(h_1^2 w_2) = 0$, contradiction, and $d_2(w_2) = \alpha \beta g$ is the only possibility.
- If $d_3(h_1 w_2) = 0$ then $d_4(h_1^2 w_2) = 0$, contradiction, and $d_3(h_1 w_2) = g^2 w_1$ is the only possibility.
**Key Differentials**

**34 to 58**

Most differentials not shown
Thank you