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Idempotents, localizations and Picard groups of $\mathcal{A}(1)$-modules

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Abstract. We analyze the stable isomorphism type of polynomial rings on degree 1 generators as modules over the subalgebra $\mathcal{A}(1) = \langle Sq^1, Sq^2 \rangle$ of the mod 2 Steenrod algebra. Since their augmentation ideals are Q_1-local, we do this by studying the Q_i-local subcategories and the associated Margolis localizations. The periodicity exhibited by such modules reduces the calculation to one that is finite. We show that these are the only localizations which preserve tensor products, by first computing the Picard groups of these subcategories and using them to determine all idempotents in the stable category of bounded-below $\mathcal{A}(1)$-modules. We show that the Picard groups of the whole category are detected in the local Picard groups, and show that every bounded-below $\mathcal{A}(1)$-module is uniquely expressible as an extension of a Q_0-local module by a Q_1-local module, up to stable equivalence.

Contents

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1. Introduction

Let H^* denote reduced mod 2 cohomology. We organize into a systematic framework the ideas that have been used to analyze the $\mathcal{A}(1)$-module structure of $H^*BV_+ = \mathbf{F}_2[x_1, \ldots, x_n]$, where V is an elementary abelian 2-group of rank n. As always, this splits into a direct sum of tensor powers of the rank 1 case, H^*BC_2. Remarkably, as an $\mathcal{A}(1)$-module, the tensor powers of H^*BC_2 are stably equivalent to their algebraic loops (syzygies). This is a general phenomenon: if I is a stably idempotent module over a finite dimensional Hopf algebra, i.e., if $I \otimes I \simeq I$, then $\Omega^n I \simeq (\Omega I)^{\otimes(n)}$:

$$\begin{aligned}(\Omega I)^{\otimes(n)} &= \Omega I \otimes \cdots \otimes \Omega I \\ &\simeq \Omega^n (I^{\otimes(n)}) \\ &\simeq \Omega^n I.\end{aligned}$$

Localizations provide a ready source of idempotents: since $\mathbf{F}_2$ is tensor idempotent, its Margolis localizations $\mathbf{L}_i\mathbf{F}_2$ are as well. It happens that $\Sigma H^*BC_2 = \Omega \mathbf{L}_1 \mathbf{F}_2$.

Our main results are as follows.

We call a bounded-below module Q_k-local if its only non-zero Margolis homology is with respect to Q_k (Definition 3.1). If M is Q_0-local then $\Omega M \simeq \Sigma M$, while if M is Q_1-local then $\Omega^4 M \simeq \Sigma^{12} M$ (Theorems 3.2 and 3.7).

We define modules R and P_0 closely related to H^*BC_2 and observe that R is Q_0-local and P_0 is Q_1-local. We show there is a unique non-split triangle

$$\Sigma R \xrightarrow{\epsilon} \mathbf{F}_2 \xrightarrow{\eta} P_0$$

(Proposition 4.2). It follows that these are Margolis localizations: $\mathbf{L}_0\mathbf{F}_2 \simeq \Sigma R$ and $\mathbf{L}_1\mathbf{F}_2 \simeq P_0$. They are therefore idempotent, and, as observed above, their tensor powers coincide with their algebraic loops, which therefore exhibit one and four-fold periodicity, respectively. Since $\Omega P_0 \simeq \Sigma H^*BC_2$, the tensor powers of H^*BC_2 exhibit four-fold periodicity. This reduces the analysis of all their tensor powers to four cases, which we carry out explicitly in Section 4.

We then deduce the basic properties of the localizations, including the fact that the natural triangle

$$\mathbf{L}_0 M \xrightarrow[\epsilon_M]{} M \xrightarrow[\eta_M]{} \mathbf{L}_1 M$$

(Definition 5.1) is the unique triangle of the form

$$M_0 \xrightarrow[\epsilon]{} M \xrightarrow[\eta]{} M_1$$

in which each M_i is Q_i-local (Theorem 5.6).

We next show that the localizations $\mathbf{L}_i\mathbf{F}_2$ and their suspensions and loops account for the whole Picard group of the Q_i-local subcategories (with no local finiteness hypotheses needed). We show that if $\mathrm{Pic}^{(i)}$ denotes the Picard group of the category of bounded-below Q_i-local modules, then

$$\begin{aligned}&\mathrm{Pic}^{(0)}(E(1)) = \mathbf{Z} &\qquad &\mathrm{Pic}^{(0)}(\mathcal{A}(1)) = \mathbf{Z}\\ &\mathrm{Pic}^{(1)}(E(1)) = \mathbf{Z} & &\mathrm{Pic}^{(1)}(\mathcal{A}(1)) = \mathbf{Z} \oplus \mathbf{Z}/(4)\end{aligned}$$

with the $\mathbf{Z}/(4)$ due to the four-fold periodicity of the loops of P_0 (Theorems 6.8 and 6.9 and Propositions 8.1 and 8.2). Next we show that the global Picard group is detected in the local ones: the localization map

$$\mathrm{Pic} \longrightarrow \mathrm{Pic}^{(0)} \oplus \mathrm{Pic}^{(1)}$$

is a monomorphism (Section 9).

We then show that the only bounded-below stably idempotent $\mathcal{A}(1)$-modules are those we have already seen (Theorem 10.1) so that we have found all localizations of the form $L(M) = I \otimes M$, I stably idempotent.

The last section in the main body of the paper observes that there is an idempotent, the Laurent series ring L, that is neither bounded-below nor bounded-above. It shows that the Margolis localizations are more fundamental than the Margolis homology: L is Q_1-local in the generalized sense that $L \simeq \mathbf{L}_1 L$ (and $\mathbf{L}_0 L \simeq 0$) despite having trivial Q_1 and Q_0 homology.

Finally, in an appendix, we give precise form to the stable equivalences we have been studying, in the expectation that these will be useful in studying the 'hit problem': the study of the $\mathcal{A}$ and $\mathcal{A}(n)$ indecomposables in H^*BV. (See [**3**], [**4**] or [**15**], for recent work on this problem.)

Since many of these results are modern versions of older results, a brief summary of their development seems in order. The algebraic loops (syzygies) of H^*BC_2 were explicitly identified in Margolis ([**10**, Chap. 23]), but had already been visible as early as the 1968 paper [**9**] by Gitler, Mahowald and Milgram, though the periodicity was not stated there. The relation to the tensor powers of H^*BC_2 was the discovery of Ossa ([**11**]). He showed that $P = H^*BC_2$ is stably idempotent as a module over the subalgebra $E(1) = E[Q_0, Q_1]$ of the Steenrod algebra, and used this to show that if V is an elementary abelian group then, modulo Bott torsion, the connective complex K-theory of BV_+ is the completion of the Rees ring of the representation ring $R(V)$ with respect to its augmentation ideal. (This is not how he said it, and his main focus was on related topological results, but this is one way of phrasing the first theorem in [**11**].) He tried to extend this to real connective K-theory, but there were flaws in his argument. By 1992, Stephan Stolz (private communication) knew that the correct statement for the real case was that $P^{\otimes(n+1)}$ was the n^{th} syzygy of P in the category of $\mathcal{A}(1)$-modules. In his unpublished 1995 Notre Dame PhD thesis, Stolz's student Cherng-Yih Yu ([**16**])) gave a proof of this together with the remarkable fact that these $\mathcal{A}(1)$-modules form the Picard group of the category of bounded-below, Q_1-local $\mathcal{A}(1)$-modules. As with Ossa's result in the complex case, this should lead to a representation theoretic description of the real connective K-theory of BV_+ modulo Bott torsion. However, this was found by other means in the author's joint work with John Greenlees ([**6**, p. 177]). More recently, Geoffrey Powell has given descriptions of the real and complex connective K-homology and cohomology of BV_+ in [**12**] and [**13**]. His functorial approach provides significant simplifications. Some of the results here are used in his work on the real case. Most recently, Shaun Ault has made use of the results here in his study [**3**] of the hit problem.

The present account is essentially self contained. In particular, we give dramatically simplified calculations of the Picard groups of the local subcategories. The work has evolved fitfully over the years since [**7**], to which it provides context and additional detail, receiving one impetus from my joint work with John Greenlees ([**5**] and [**6**]), another from questions asked by Vic Snaith (which led to [**8**]), and a more recent one from discussions with Geoffrey Powell in connection with [**13**]. I am grateful to Geoffrey Powell for many useful discussions while working out some of these results and to the University of Paris 13 for the opportunity to work on this in May of 2012.

2. Recollections

We begin with some basic definitions and results about modules over finite sub-Hopf algebras of the mod 2 Steenrod algebra, in order to state clearly the hypotheses under which they hold. The reader who is familiar with $\mathcal{A}(1)$-modules should probably skip to the next section.

Let $\mathcal{A}(n)$ be the subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ generated by $\{Sq^{2^i} \mid 0 \leq i \leq n\}$. Thus $\mathcal{A}(0)$ is exterior on one generator, Sq^1, and $\mathcal{A}(1)$, generated by Sq^1 and Sq^2, is 8 dimensional.

Let $E(n)$ be the exterior subalgebra of $\mathcal{A}$ generated by the Milnor primitives $\{Q_i \mid 0 \leq i \leq n\}$. (Recall that $Q_0 = Sq^1$ and $Q_n = Sq^{2^n} Q_{n-1} + Q_{n-1} Sq^{2^n}$.) $E(n)$ is a sub-Hopf algebra of $\mathcal{A}(n)$.

For $B = E(1)$, $\mathcal{A}(1)$, or any finite sub-Hopf algebra of $\mathcal{A}$, let B-Mod be the category of all graded B-modules. The category B-Mod is abelian, complete, cocomplete, has enough projectives and injectives, and has a symmetric monoidal product $\otimes = \otimes_{\mathbf{F}_2}$. Since B is a Frobenius algebra, free, projective and injective are equivalent conditions in B-Mod. (See Margolis ([**10**]), Chapters 12, 13 and 15, and in particular Lemma 15.27 for details.)

The best results hold in the abelian subcategory B-Mod^b of bounded-below B-modules. It has enough projectives and injectives ([**10**, Lemma 15.27]). A module in B-Mod^b is free, projective, or injective there iff it is so in B-Mod ([**10**, Lemma 15.17]).

Since the algebras B we are considering are Poincare duality algebras, the following decomposition result holds without restriction on M. It will be useful in our discussion of stable isomorphism.

Proposition 2.1 ([**10**, Proposition 13.13 and p. 203]). *A module M in B-Mod has an expression*

$$M \cong F \oplus M^{\mathrm{red}},$$

unique up to isomorphism, where F is free and M^{red} has no free summands.

Definition 2.2. We call M^{red} the *reduced* part of M.

Note that we are not asserting that $M \mapsto M^{\mathrm{red}}$ is a functor, or that there are *natural* maps $M \longrightarrow M^{\mathrm{red}}$ or $M^{\mathrm{red}} \longrightarrow M$.

Definition 2.3. If $\mathcal{C}$ is a subcategory of B-Mod which contains the projective modules, the *stable module category of* $\mathcal{C}$, written $\mathrm{St}(\mathcal{C})$, is the category with the same objects as $\mathcal{C}$ and with morphisms replaced by their equivalence classes modulo those which factor through a projective module. Let us write $M \simeq N$ to denote *stable isomorphism*, isomorphism in $\mathrm{St}(B\text{-Mod})$, and reserve $M \cong N$ for isomorphism in B-Mod.

Over a finite Hopf algebra like B, stable isomorphism simplifies.

Proposition 2.4 ([**10**, Proposition 14.1]). *In B-Mod, modules M and N are stably isomorphic iff there exist free modules P and Q such that $M \oplus P \cong N \oplus Q$.*

In B-Mod^b, stable isomorphism simplifies further.

PROPOSITION 2.5 ([**10**, Proposition 14.11]). *Let M and N be modules in B-Mod^b.*

(1) *$M \simeq N$ iff $M^{\mathrm{red}} \cong N^{\mathrm{red}}$.*
(2) *$f : M \longrightarrow N$ is a stable equivalence iff*

$$M^{\mathrm{red}} \rightarrowtail M \xrightarrow{f} N \twoheadrightarrow N^{\mathrm{red}}$$

is an isomorphism in B-Mod.

Here, $M^{\mathrm{red}} \rightarrowtail M$ and $N \twoheadrightarrow N^{\mathrm{red}}$ are any maps which are part of a splitting of M and N, respectively, into a free summand and a reduced summand.

The preceding result holds for all finite Hopf algebras. For modules over subalgebras B of the mod 2 Steenrod algebra, the theorem of Adams and Margolis ([**1**] or [**10**, Theorem 19.6]) gives us a simple criterion for stable isomorphism in B-Mod^b. Recall that the Milnor primitives Q_i satisfy $Q_i^2 = 0$, so that we may define $H(M, Q_i) = \mathrm{Ker}(Q_i)/\mathrm{Im}(Q_i)$.

THEOREM 2.6. *Let $B = \mathcal{A}(1)$ or $E(1)$. Suppose that $f : M \longrightarrow N$ in B-Mod^b. If f induces isomorphisms $f_* : H(M, Q_i) \longrightarrow H(N, Q_i)$ for $i = 0$ and $i = 1$, then f is a stable isomorphism.*

In particular, if a bounded-below module M has trivial Q_0 and Q_1 homology, then the map $0 \longrightarrow M$ is a stable equivalence, and therefore M is free.

REMARK 2.7. The hypothesis that the modules be bounded-below is needed for Theorem 2.6 to hold: the Laurent series ring $\mathbf{F}_2[x, x^{-1}]$ is not free over $E(1)$ or $\mathcal{A}(1)$, yet has trivial Q_0 and Q_1 homology.

Margolis ([**10**, Theorem 19.6.(b)]) gives a similar characterization of stable isomorphism or modules over any sub-Hopf algebra B of the mod 2 Steenrod algebra.

Finally, we consider the algebraic loops functor. By Schanuel's Lemma, letting ΩM be the kernel of an epimorphism from a projective module to M gives a well defined module up to stable isomorphism. To get functoriality, the following definition is simplest.

DEFINITION 2.8. Let $I = \mathrm{Ker}(B \longrightarrow \mathbf{F}_2)$ be the augmentation ideal of B. Let $\Omega M = I \otimes M$.

Note that $\Omega\mathbf{F}_2 \cong I$. Similarly, we may define the inverse loops functor.

DEFINITION 2.9. Let $I^{-1} = \mathrm{Cok}(\mathbf{F}_2 \longrightarrow \Sigma^{-d}B))$ be the cokernel of the d^{th} desuspension of the the inclusion of the socle into B. (d is 4 if $B = E(1)$, 6 if $B = \mathcal{A}(1)$.) Let $\Omega^{-1}M = I^{-1} \otimes M$.

To see that the notation makes sense, recall the 'untwisting' isomorphism

$$\theta : B \otimes M \longrightarrow B \otimes \widehat{M},$$

given by $\theta(b \otimes m) = \sum b' \otimes b'' m$. Here $B \otimes \widehat{M}$ is the free B-module on the underlying vector space $\widehat{M}$ of M and $\psi(b) = \sum b' \otimes b''$ is the coproduct of b. The inverse, $\theta^{-1}(b \otimes m) = \sum b' \otimes \chi(b'')m$, where χ is the conjugation (antipode) of B. This shows that tensoring with a free module gives a free module.

In particular, tensor product is well defined in the stable module category.

Tensoring the short exact sequence $0 \longrightarrow I \longrightarrow B \longrightarrow \mathbf{F}_2 \longrightarrow 0$ with I^{-1} shows that $I \otimes I^{-1}$ is stably equivalent to $\mathbf{F}_2$.

Corollary 2.10. *We have stable equivalences* $\Omega\Omega^{-1} \simeq \mathrm{Id} \simeq \Omega^{-1}\Omega$. *In general,* $\Omega^k\Omega^l \simeq \Omega^{k+l}$ *for all integers* k *and* l.

Finally, we should note that the stable module category is triangulated. For any short exact sequence of modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

there is an extension cocycle $\Omega M_3 \longrightarrow M_1$ (or equivalently $M_3 \longrightarrow \Omega^{-1}M_1$) representing the extension class in $\mathrm{Ext}^1_B(M_3, M_1)$. The triangles in the stable module category are the sequences

$$\Omega M_3 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

and

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \Omega^{-1}M_1.$$

for the short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

3. Periodicity

We start by observing the periodicities which *local* B-modules obey, for $B = E(1)$ or $\mathcal{A}(1)$. We shall restrict attention to the category $B\text{-Mod}^b$ of bounded-below B-modules.

Definition 3.1. Let B be either $E(1)$ or $\mathcal{A}(1)$. Call a B-module Q_k-*local* if $H(M, Q_i) = 0$ for $i \neq k$. For $k \in \{0, 1\}$, let $B\text{-Mod}^{(k)}$ be the full subcategory of $B\text{-Mod}^b$ containing the Q_k-local modules.

Theorem 3.2. *If* $M \in B\text{-Mod}^{(0)}$ *then* $\Omega M \simeq \Sigma M$.

Proof. Evidently, $\mathcal{A}(0)$ has a unique B-module (even, $\mathcal{A}$-module) structure compatible with its structure as a module over itself. Tensor M with the short exact sequence of B-modules

$$0 \longrightarrow \Sigma\mathbf{F}_2 \longrightarrow \mathcal{A}(0) \longrightarrow \mathbf{F}_2 \longrightarrow 0.$$

We obtain

$$0 \longrightarrow \Sigma M \longrightarrow M \otimes \mathcal{A}(0) \longrightarrow M \longrightarrow 0.$$

By Theorem 2.6 and the Künneth isomorphism for Q_i homology, the module in the middle is free and the result follows. □

The Q_1-local case requires a bit of preparation. Recall the notation $A/\!\!/B$ for the A-module $A \otimes_B \mathbf{F}_2$ when B is a sub-(Hopf-)algebra of A.

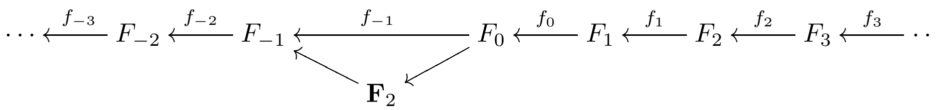
Definition 3.3. Define modules F_i and maps $f_i : F_{i+1} \longrightarrow F_i$ for $i \in \mathbf{Z}$ by $F_{i+4} = \Sigma^{12}F_i$, $f_{i+4} = \Sigma^{12}f_i$, $f_3 = Sq^2Sq^3$ and the following:

$$\begin{array}{ccccccccccccccc}
0 & \longleftarrow & \mathbf{F}_2 & \longleftarrow & F_0 & \xleftarrow{f_0} & F_1 & \xleftarrow{f_1} & F_2 & \xleftarrow{f_2} & F_3 & \longleftarrow & \Sigma^{12}\mathbf{F}_2 & \longleftarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \| & & \| & & \\
0 & \longleftarrow & \mathbf{F}_2 & \longleftarrow & \mathcal{A}(1)/\!\!/\mathcal{A}(0) & \xleftarrow{Sq^2} & \Sigma^2\mathcal{A}(1) & \xleftarrow{Sq^2} & \Sigma^4\mathcal{A}(1) & \xleftarrow{Sq^3} & \Sigma^7\mathcal{A}(1)/\!\!/\mathcal{A}(0) & \xleftarrow{Sq^2Sq^3} & \Sigma^{12}\mathbf{F}_2 & \longleftarrow & 0
\end{array}$$

The following is an elementary calculation, originally due to Toda [**14**]. The diagram in the proof of Proposition 3.6 is sufficient to prove it.

Proposition 3.4. *The sequence in Definition 3.3 is exact.* □

Splicing this sequence and its suspensions, we obtain a complete (i.e., Tate) resolution of $\mathbf{F}_2$ by modules tensored up from $\mathcal{A}(0)$: the F_{4i} and F_{4i+3} are suspensions of $\mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbf{F}_2$, while the F_{4i+1} and F_{4i+2} are suspensions of $\mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathcal{A}(0)$.



The cokernels in this sequence will play an important role. They are the syzygies of $\mathbf{F}_2$ with respect to the relative projective class of projectives relative to the $\mathcal{A}(0)$-split exact sequences.

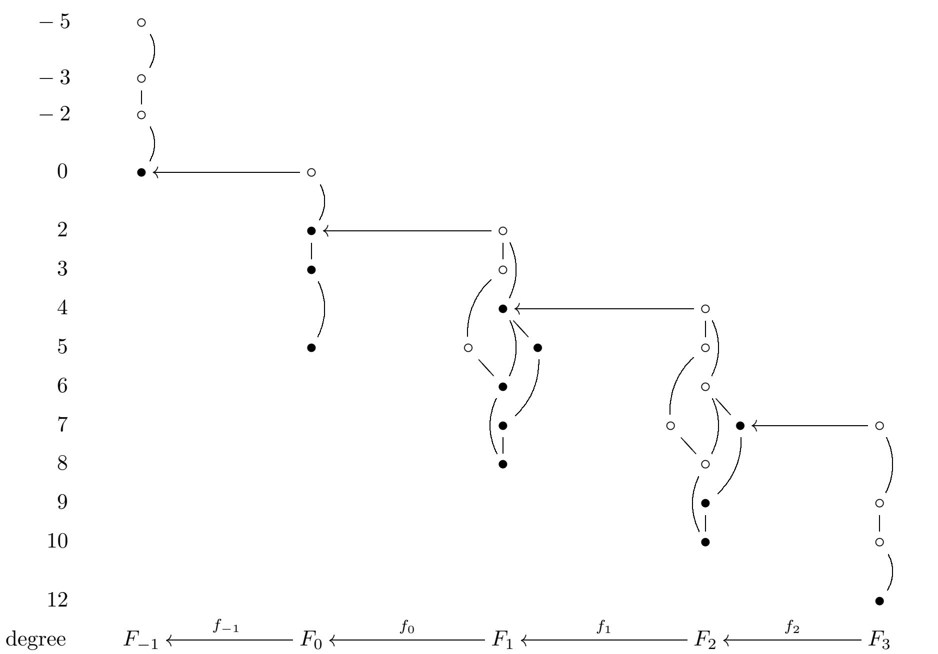
Definition 3.5. Let $M_i = \Sigma^{-i}\mathrm{Cok} f_i$.

We have inserted the suspension here to make later calculations run more smoothly. It is a simple matter to describe the M_i.

Proposition 3.6. *For each $i \in \mathbf{Z}$, $M_{i+4} = \Sigma^8 M_i$, so the following suffice to determine all the M_i:*

- $M_0 = \mathbf{F}_2$,
- $M_1 = \Sigma\mathcal{A}(1)/(Sq^2)$,
- $M_2 = \Sigma^2\mathcal{A}(1)/(Sq^3)$,
- $M_3 = \Sigma^4\mathcal{A}(1)/(Sq^1, Sq^2Sq^3)$.

Proof. The following diagram exhibits the $\Sigma^i M_i$ by open dots in the diagram of F_i, or as solid dots in the diagram of F_{i-1}.



□

THEOREM 3.7. *If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$, then $\Omega^i M \simeq M_i \otimes M$. In particular, $\Omega^{i+4} M \simeq \Sigma^{12}\Omega^i M$.*

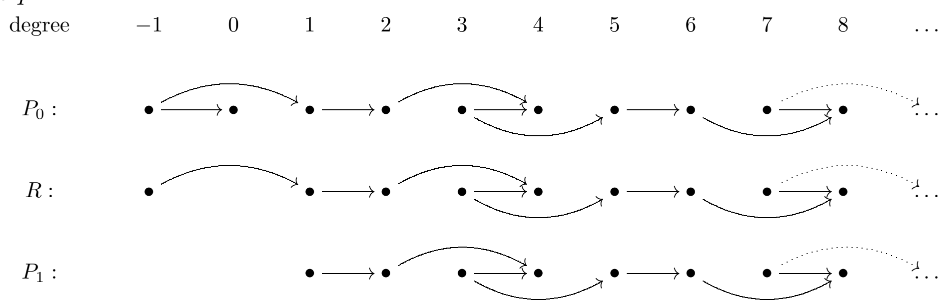
PROOF. The modules F_i have no Q_1 homology, while M has only Q_1 homology. Therefore, the $F_i \otimes M$ are $\mathcal{A}(1)$-free by the Künneth isomorphism and Theorem 2.6. Since $M_0 \otimes M \cong M$, the result follows from exactness of the sequence of $F_i \otimes M$. □

4. Reduction from $P_1^{\otimes(n)}$ to $\Omega^n P_1$

In this section we introduce the Q_i-localizations of $\mathbf{F}_2$ and determine some of their basic properties. As a corollary, we will obtain the stable isomorphism type of H^*BV for elementary abelian 2-groups V.

DEFINITION 4.1. Let $P_1 = H^*BC_2 = (x)$, the ideal generated by x in $H^*BC_{2+} = \mathbf{F}_2[x]$. Let P_0 be the submodule of the Laurent series ring $L = \mathbf{F}_2[x, x^{-1}]$ which is nonzero in degrees -1 and higher. Let R be the quotient of the unique inclusion $\eta : \mathbf{F}_2 \longrightarrow P_0$. Let $\epsilon : \Sigma R \longrightarrow \mathbf{F}_2$ be the unique non-zero homomorphism.

We represent P_0, P_1 and R diagrammatically by showing the action of Sq^1 and Sq^2:



We record some obvious facts using the results of the preceding section.

PROPOSITION 4.2. *The following hold.*

- *The module R is Q_0-local, and the modules P_0 and P_1 are Q_1-local.*
- *There are short exact sequences*

$$0 \longrightarrow \Sigma P_1 \longrightarrow \Sigma R \xrightarrow{\epsilon} \mathbf{F}_2 \longrightarrow 0$$

  and

$$0 \longrightarrow \mathbf{F}_2 \xrightarrow{\eta} P_0 \longrightarrow R \longrightarrow 0.$$

- *ϵ is the extension cocycle for the second of these exact sequences, giving a triangle*

$$\Sigma R \xrightarrow{\epsilon} \mathbf{F}_2 \xrightarrow{\eta} P_0$$

  in $\mathrm{St}(\mathcal{A}(1)\text{-Mod})$.

PROOF. All but the last item are clear from inspection. If we let $F = R \otimes \mathcal{A}(0)$, then, as in the proof of Theorem 3.2, F is $\mathcal{A}(1)$-free and lies in a short exact sequence $0 \longrightarrow \Sigma R \longrightarrow F \longrightarrow R \longrightarrow 0$. The epimorphism $F \longrightarrow R$ lifts to P_0, yielding a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{F}_2 & \xrightarrow{\eta} & P_0 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \epsilon} & & \uparrow & & \| & & \\
0 & \longrightarrow & \Sigma R & \longrightarrow & F & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

□

We will see in Section 5 that the map η is the Q_1-localization of $\mathbf{F}_2$, with corresponding Q_1-nullification ϵ. Dually, ϵ is the Q_0-colocalization of $\mathbf{F}_2$ with corresponding Q_0-conullification η. As noted in the introduction, it follows that if $I = P_0$ or $I = \Sigma R$ then I is idempotent, and that therefore $\Omega^n I \simeq (\Omega I)^{\otimes(n)}$. This underlies the argument which we now use to produce minimal representatives for the tensor powers of H^*BC_2.

THEOREM 4.3. *If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ then $\Omega M \simeq \Sigma P_1 \otimes M$ and $\eta \otimes 1$ is a stable equivalence $M \xrightarrow{\simeq} P_0 \otimes M$. In particular, for $n \geq 1$, $P_1^{\otimes(n)} \simeq \Sigma^{-n}\Omega^n P_0$. If $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ then $\epsilon \otimes 1$ is a stable equivalence $\Sigma R \otimes M \xrightarrow{\simeq} M$.*

PROOF. If M is Q_1-local, then $R \otimes M$ has trivial Q_i-homology for both $i = 0$ and $i = 1$. If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ then $R \otimes M$ is also bounded-below, and hence free by Theorem 2.6. Tensoring the first short exact sequence of the preceding proposition with M then gives that $\Omega M \simeq \Sigma P_1 \otimes M$. Tensoring the second one with M shows that $\eta \otimes 1$ is a stable equivalence.

Since P_0 and P_1 are in $\mathcal{A}(1)\text{-Mod}^{(1)}$, we have $\Omega P_0 \simeq \Sigma P_1 \otimes P_0 \simeq \Sigma P_1$, proving the $n = 1$ case of the equivalence $P_1^{\otimes(n)} \simeq \Sigma^{-n}\Omega^n P_0$. The remaining cases then follow immediately by induction:

$$\begin{aligned}\Omega^{n+1}P_0 = \Omega\Omega^n P_0 &\simeq \Omega\Sigma^n P_1^{\otimes(n)} \\ &\simeq \Sigma P_1 \otimes \Sigma^n P_1^{\otimes(n)} \cong \Sigma^{n+1}P_1^{\otimes(n+1)}\end{aligned}$$

The last statement is proved dually: since $\Sigma P_1 \otimes M$ is free by the Künneth isomorphism and Theorem 2.6, $\epsilon \otimes 1$ is a stable equivalence. □

Determining minimal representatives for the tensor powers $P_1^{\otimes n}$ is now reduced to finding minimal representatives for the $\Omega^n P_0$. By periodicity, we only need the first four. The following definition will be convenient.

DEFINITION 4.4. For $n \in \mathbf{Z}$, let $P_n = (\Sigma^{-n}\Omega^n P_0)^{\text{red}}$.

Clearly the notation is consistent with our definitions of P_i, $i = 0, 1$. We first record some obvious facts.

THEOREM 4.5. *The modules P_n are Q_1-local and satisfy the following equivalences.*

- *If $n \geq 1$ then $(P_1^{\otimes(n)})^{\text{red}} = P_n$.*
- *$P_{n+4} \cong \Sigma^8 P_n$,*
- *$P_n \otimes P_m \simeq P_{n+m}$, and*
- *$\Omega P_n \simeq \Sigma P_{n+1}$.*

PROOF. The first statement is immediate from the definition of P_n and Theorem 4.3. Since $\Omega^{n+4}P_0 \simeq \Sigma^{12}\Omega^n P_0$ by Proposition 4.2 and Theorem 3.7, we have a stable equivalence $P_{n+4} \simeq \Sigma^8 P_n$. But, both sides are reduced and hence they are isomorphic (Theorem 2.5). The third and fourth statements are immediate consequences of the first. □

The modules M_i which appear in the sequence of Proposition 3.4 (Definition 3.5) all occur as submodules of the P_i. See Figure 1 for diagrammatic representations of them.

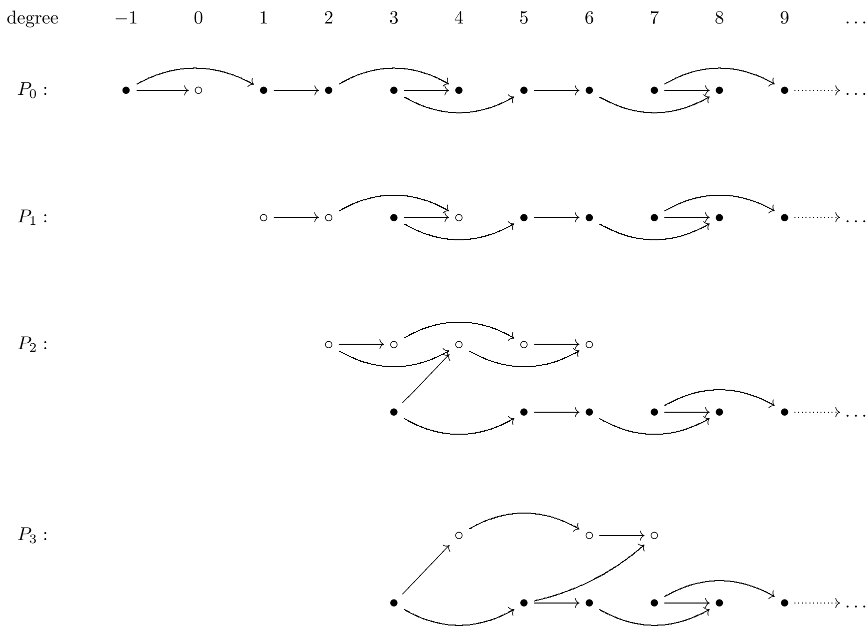


FIGURE 1. The modules P_n, $0 \leq n \leq 3$. The submodules M_n are indicated by open dots ($\circ$).

THEOREM 4.6. *There are short exact sequences*

$$0 \longrightarrow M_0 \longrightarrow P_0 \longrightarrow R \longrightarrow 0$$

$$0 \longrightarrow M_1 \longrightarrow P_1 \longrightarrow \Sigma^4 R \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow P_2 \longrightarrow \Sigma^4 R \longrightarrow 0$$

$$0 \longrightarrow M_3 \longrightarrow P_3 \longrightarrow \Sigma^4 R \longrightarrow 0$$

Each of these is the unique non-trivial extension, with Sq^1 of the bottom class in the suspension of R equal to the unique element of M_i of the relevant degree.

PROOF. The first short exact sequence is a restatement of the last short exact sequence in Proposition 4.2. Next, the submodule of P_1 generated by the bottom class is M_1 and the quotient by it is $\Sigma^4 R$. This gives

$$0 \longrightarrow M_1 \longrightarrow P_1 \longrightarrow \Sigma^4 R \longrightarrow 0, \tag{4.1}$$

the second of our claimed short exact sequences. Taking minimal free modules mapping onto the three modules in (4.1) and applying the snake lemma produces the suspension of the next of our claimed sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & P_1 & \longrightarrow & \Sigma^4 R & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma\mathcal{A}(1) & \longrightarrow & F_1 & \longrightarrow & \Sigma^4 F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma M_2 & \longrightarrow & \Sigma P_2 & \longrightarrow & \Sigma^5 R & \longrightarrow & 0
\end{array}$$

Here $F_1 = \Sigma\mathcal{A}(1) \oplus \Sigma^4 F$ and F is the free module used in the proof of Proposition 4.2. It is easy to check that the top and bottom rows in the preceding diagram are each the unique non-trivial extension.

Applying this procedure again, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_2 & \longrightarrow & P_2 & \longrightarrow & \Sigma^4 R & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma^2\mathcal{A}(1) & \longrightarrow & F_2 & \longrightarrow & \Sigma^4 F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma^9 DM_1 & \longrightarrow & \Sigma P_3 & \longrightarrow & \Sigma^5 R & \longrightarrow & 0
\end{array}$$

Here $F_2 = \Sigma^2\mathcal{A}(1) \oplus \Sigma^4 F$ and $DM = \mathrm{Hom}(M, \mathbf{F}_2)$ is the dual of M. Removing one suspension gives the last of our short exact sequences. □

REMARK 4.7. In [**11**], Lemma 2 asserts that $P_1 \otimes P_1$ is stably equivalent to $\Sigma^2 P_1$ rather than $\Sigma^{-1}\Omega P_1$. These are the same in the category of $E(1)$-modules, but not in the category of $\mathcal{A}(1)$-modules. These modules differ by one copy of $E(1)$. This also makes Proposition 2 there false, both in identifying the degrees of the Eilenberg-MacLane summands, and in identifying the complement to them. See Corollary B.4 for the correct statement.

5. Q_i-local $A(1)$-modules

Again let $B = E(1)$ or $\mathcal{A}(1)$. We now consider the two Margolis localizations (at Q_0 and at Q_1) of B-Mod.

DEFINITION 5.1. Let $\epsilon : \Sigma R \longrightarrow \mathbf{F}_2$ and $\eta : \mathbf{F}_2 \longrightarrow P_0$ be the unique non-zero homomorphisms. Define functors $\mathbf{L}_i : B\text{-Mod}^b \longrightarrow B\text{-Mod}^{(i)}$ and natural transformations $\eta_M : M \longrightarrow \mathbf{L}_1 M$ and $\epsilon_M : \mathbf{L}_0 M \longrightarrow M$ by

$$\begin{array}{ccccc}
\Sigma R \otimes M & \xrightarrow{\epsilon\otimes 1} & \mathbf{F}_2 \otimes M & \xrightarrow{\eta\otimes 1} & P_0 \otimes M \\
\Big\| & & \Big\downarrow \cong & & \Big\| \\
\mathbf{L}_0 M & \xrightarrow[\epsilon_M]{} & M & \xrightarrow[\eta_M]{} & \mathbf{L}_1 M
\end{array}$$

The functors they induce on stable module categories are idempotent, orthogonal, semi-ring homomorphisms. We make these statements precise as follows.

THEOREM 5.2. *$\mathbf{L}_i M$ is Q_i-local. $\mathbf{L}_0$ and $\mathbf{L}_1$ are exact and additive, and preserve tensor products up to stable equivalence.*

(1) $\mathbf{L}_0\mathbf{L}_1 M \simeq 0 \simeq \mathbf{L}_1\mathbf{L}_0 M$.
(2) ϵ_M *induces an isomorphism of* Q_0 *homology.*
(3) η_M *induces an isomorphism of* Q_1 *homology.*
(4) *If* $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ *then* ϵ_M *is a stable equivalence and* $\mathbf{L}_1 M \simeq 0$.
(5) *If* $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ *then* η_M *is a stable equivalence and* $\mathbf{L}_0 M \simeq 0$.

PROOF. That $\mathbf{L}_i M$ is Q_i-local is immediate from the Künneth theorem for Q_j homology and Proposition 4.2. It is a general fact that tensor product is exact and preserves direct sums. Applying Theorem 4.3 to $M = R$ and $M = P_0$, we find that $\Sigma R \otimes \Sigma R \simeq \Sigma R$ and $P_0 \otimes P_0 \simeq P_0$. Preservation of tensor products then follows by associativity and this idempotence. Statement (1) follows from the fact that $\Sigma R \otimes P_0$ is free by Proposition 4.2, the Künneth theorem, and Theorem 2.6. Then (2) and (3) follow from the Künneth theorem for Q_i homology and the case $M = \mathbf{F}_2$. Finally, (4) and (5) are then immediate by the theorem of Adams and Margolis (Theorem 2.6). □

Here is a more precise form of idempotence.

THEOREM 5.3. *The* $\mathbf{L}_i$ *are stably idempotent. In particular, the following hold.*

(1) $\mathbf{L}_0\epsilon_M$, $\epsilon_{\mathbf{L}_0 M}$, $\mathbf{L}_1\eta_M$, *and* $\eta_{\mathbf{L}_1 M}$ *are stable equivalences.*
(2) $\mathbf{L}_0\epsilon_M \simeq \epsilon_{\mathbf{L}_0 M}$ *and* $\mathbf{L}_1\eta_M \simeq \eta_{\mathbf{L}_1 M}$.
(3) $\mathbf{L}_0\epsilon_M$ *and* $\epsilon_{\mathbf{L}_0 M}$ *are not equal, but are coequalized by* ϵ_M.
(4) $\mathbf{L}_1\eta_M$ *and* $\eta_{\mathbf{L}_1 M}$ *are not equal, but are equalized by* η_M.

PROOF. Statement (1) is immediate from the preceding Theorem. Statements (3) and (4) are elementary calculations: $\epsilon \otimes 1$ and $1 \otimes \epsilon$ are coequalized by ϵ, while $\eta \otimes 1$ and $1 \otimes \eta$ are equalized by η. To show the stable equivalences in (2), it suffices to treat the case $M = \mathbf{F}_2$. For this, we use Proposition 2.5. Since $(\Sigma R \otimes \Sigma R)^{\text{red}} \cong \Sigma R$, we need a stable equivalence $\Sigma R \longrightarrow \Sigma R \otimes \Sigma R$ which equalizes $\epsilon \otimes 1$ and $1 \otimes \epsilon$. We define such an $\mathcal{A}(1)$ homomorphism by

$$i(\Sigma x^n) = \sum_{i+j=n-1} \Sigma x^i \otimes \Sigma x^j$$

where we treat Σx^0 as zero, and let i and j range over integers ≥ -1. It is immediate that $(\epsilon \otimes 1)i = (1 \otimes \epsilon)i$ so that $\epsilon \otimes 1 \simeq 1 \otimes \epsilon$.

Dually, for the stable equivalence between $\eta \otimes 1$ and $1 \otimes \eta$, we observe that they are coequalized by the stable equivalence $P_0 \otimes P_0 \longrightarrow P_0$ given by

$$x^i \otimes x^j \mapsto \begin{cases} 0 & i \equiv -1 \pmod 4 \quad \text{and} \quad j \equiv -1 \pmod 2 \\ x^{i+j} & \text{otherwise.} \end{cases}$$ □

PROPOSITION 5.4. *Algebraic loops commute with the* $\mathbf{L}_i$*:* $\Omega\mathbf{L}_i M \simeq \mathbf{L}_i\Omega M$. *In addition,*

(1) $\Omega\mathbf{L}_0 M \simeq \Sigma\mathbf{L}_0 M$
(2) $\Omega^i\mathbf{L}_1 M \simeq \Sigma^i P_i \otimes M$
(3) $\Omega^4\mathbf{L}_1 M \simeq \mathbf{L}_1\Omega^4 M \simeq \Sigma^{12}\mathbf{L}_1 M$

PROOF. Since tensoring a B-module with a free B-module gives a free B-module, we have

$$\Omega(M \otimes N) \simeq (\Omega M) \otimes N \simeq M \otimes (\Omega N).$$

Then $\Omega\mathbf{L}_0 M = \Omega(\Sigma R \otimes M) \simeq \Sigma^2 R \otimes M$, proving (1). Similarly (2) follows because $\Omega^i \mathbf{L}_1 M = \Omega^i(P_0 \otimes M) \simeq (\Omega^i P_0) \otimes M \simeq \Sigma^i P_i \otimes M$ by Theorem 4.5. Then (3) follows since $\Sigma^4 P_4 = \Sigma^{12} P_0$. □

PROPOSITION 5.5. *There are short exact sequences*

$$0 \longrightarrow M \xrightarrow{\eta_M} \mathbf{L}_1(M) \longrightarrow \Sigma^{-1}\mathbf{L}_0(M) \longrightarrow 0$$

and

$$0 \longrightarrow \Omega\mathbf{L}_1(M) \longrightarrow \mathbf{L}_0(M) \xrightarrow{\epsilon_M} M \longrightarrow 0.$$

PROOF. These follow from the short exact sequences of modules

$$0 \longrightarrow \mathbf{F}_2 \xrightarrow{\eta} P_0 \longrightarrow R \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma P_1 \longrightarrow \Sigma R \xrightarrow{\epsilon} \mathbf{F}_2 \longrightarrow 0.$$ □

THEOREM 5.6. *Each $M \in St(B\text{-Mod}^b)$ sits in a unique triangle $M_0 \longrightarrow M \longrightarrow M_1$ with $M_i \in B\text{-Mod}^{(i)}$. Therefore, a B-module in $B\text{-Mod}^b$ is uniquely determined, up to stable equivalence, by a triple $(M_0, M_1, e(M))$, where $M_i \in B\text{-Mod}^{(i)}$ and $e(M) \in \mathrm{Ext}_B^{1,0}(M_1, M_0)$.*

PROOF. The diagram

$$\begin{array}{ccccc}
\mathbf{L}_0 M_0 & \xrightarrow{\simeq} & M_0 & \longrightarrow & \mathbf{L}_1 M_0 \simeq 0 \\
\downarrow \simeq & & \downarrow & & \downarrow \\
\mathbf{L}_0 M & \xrightarrow{\epsilon_M} & M & \xrightarrow{\eta_M} & \mathbf{L}_1 M \\
\downarrow & & \downarrow & & \downarrow \simeq \\
0 \simeq \mathbf{L}_0 M_1 & \longrightarrow & M_1 & \xrightarrow{\simeq} & \mathbf{L}_1 M_1
\end{array}$$

shows that the triangle $M_0 \longrightarrow M \longrightarrow M_1$ is equivalent to the canonical one, $\mathbf{L}_0 M \longrightarrow M \longrightarrow \mathbf{L}_1 M$. □

REMARK 5.7. Finally, it is clear that we can extend these definitions to all B-modules. The fundamental triangle

$$\mathbf{L}_0 M \xrightarrow{\epsilon_M} M \xrightarrow{\eta_M} \mathbf{L}_1 M$$

then implies that a homomorphism $f : M \longrightarrow N$ in B-Mod is a stable equivalence iff both $\mathbf{L}_0(f)$ and $\mathbf{L}_1(f)$ are stable equivalences. It shows, in particular, that M is free iff both $\mathbf{L}_0 M$ and $\mathbf{L}_1 M$ are free.

This criterion for equivalence is the same as that of Adams and Margolis for bounded below modules, but holds in full generality. The example of Section 11 shows that this is a genuine generalization.

6. Pic and Pic$^{(k)}$

Again let B be either $E(1)$ or $\mathcal{A}(1)$ and $\mathrm{St}(\mathcal{C})$ the stable category of a subcategory $\mathcal{C}$ of B-Mod (Definition 2.3). Since $\mathbf{F}_2$ is the unit for tensor product in B-Mod, its localizations $\mathbf{L}_i\mathbf{F}_2$ are the units for tensor product in the local subcategories.

Proposition 6.1. *ΣR is the unit for tensor product in* $\mathrm{St}(B\text{-Mod}^{(0)})$ *and P_0 is the unit for tensor product in* $\mathrm{St}(B\text{-Mod}^{(1)})$. *The stable equivalence classes of modules $M \in St(B\text{-Mod}^{(k)})$ or $\mathrm{St}(B\text{-Mod}^{b})$ form a (possibly big) semi-ring with unit under direct sum and tensor product.* □

The Picard groups are the multiplicative groups in these semi-rings.

Definition 6.2. Let

- $\widetilde{\mathrm{Pic}}(B) = \Big(\mathrm{Obj}(\mathrm{St}(B\text{-Mod}^{b}))/\simeq\Big)^{\times}$ and
- $\mathrm{Pic}^{(k)}(B) = \Big(\mathrm{Obj}(\mathrm{St}(B\text{-Mod}^{(k)}))/\simeq\Big)^{\times}$.

Let $\mathrm{Pic}(B)$ be the subgroup of $\widetilde{\mathrm{Pic}}(B)$ whose elements are represented by finitely generated modules.

Remark 6.3. Of these, only $\mathrm{Pic}(B)$ is clearly a set. We will show, by explicitly calculating them, that $\mathrm{Pic}^{(0)}(B)$ and $\mathrm{Pic}^{(1)}(B)$ are sets. It would be interesting to know whether $\mathrm{Pic}(B) = \widetilde{\mathrm{Pic}}(B)$, and, if not, how much larger $\widetilde{\mathrm{Pic}}(B)$ is.

Adams and Priddy characterize the elements in $\mathrm{Pic}(B)$. It is pertinent to recall that $H(M, Q_k)$ depends only upon the stable isomorphism type of M.

Lemma 6.4 ([**2**, Lemma 3.5]). *$M \in \mathrm{Pic}(B)$ iff each $H(M, Q_k)$ is one dimensional.*

Adams and Priddy remark that, if one drops the hypothesis of finite generation, then having $H(M, Q_k)$ one dimensional for each k no longer implies that M is invertible. The module $P_0 \oplus \Sigma R$ is an example. The other direction does hold in general, though.

Lemma 6.5. *If $M \in \mathrm{Pic}^{(k)}(B)$ then $H(M, Q_k)$ is one dimensional.*

The converse, Corollary 8.3, will follow from the calculations of $\mathrm{Pic}^{(k)}$, since those calculations will show that if $M \in B\text{-Mod}^{(k)}$ and $H(M, Q_k)$ is one dimensional, then M is stably isomorphic to an invertible module.

After characterizing the invertible B-modules, Adams and Priddy go on to compute $\mathrm{Pic}(E(1))$ and $\mathrm{Pic}(\mathcal{A}(1))$.

Theorem 6.6 ([**2**, Theorem 3.6]). *$\mathrm{Pic}(E(1)) = \mathbf{Z} \oplus \mathbf{Z}$, generated by $\Sigma\mathbf{F}_2$ and the augmentation ideal $\Omega\mathbf{F}_2 = \mathrm{Ker}(E(1) \longrightarrow \mathbf{F}_2)$.*

Theorem 6.7 ([**2**, Theorem 3.7]). *$\mathrm{Pic}(\mathcal{A}(1)) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/(2)$, generated by $\Sigma\mathbf{F}_2$, the augmentation ideal $\Omega\mathbf{F}_2 = \mathrm{Ker}(\mathcal{A}(1) \longrightarrow \mathbf{F}_2)$, and $\Sigma^{-4}M_2$.*

The module $J = \Sigma^{-4}M_2$ is known as the 'joker' for its role as a torsion element in $\mathrm{Pic}(\mathcal{A}(1))$ and for the resemblance of its diagrammatic depiction (Figure 1) to a traditional jester's hat.

We now turn to the determination of the local Pic groups. In his thesis ([**16**]), Cherng-Yih Yu computed $\mathrm{Pic}^{(1)}$ for both $E(1)$ and $\mathcal{A}(1)$. His calculation of $\mathrm{Pic}^{(1)}(E(1))$ is easy, and we give it now. His calculation of $\mathrm{Pic}^{(1)}(\mathcal{A}(1))$ is quite complicated and computational. In the next section, we give a simpler and more straightforward calculation of it. Following that, we compute $\mathrm{Pic}^{(0)}$ for both $E(1)$ and $\mathcal{A}(1)$.

Recall that, as an $E(1)$-module, $P_i \simeq \Sigma^{2i}P_0$ (see Remark 4.7).

THEOREM 6.8 ([**16**, Lemma 2.5]). *If $M \in E(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \Sigma^s \mathbf{F}_2$ then $M \simeq \Sigma^s P_0$. Therefore, $\mathrm{Pic}^{(1)}(E(1)) = \{\Sigma^i P_0\} \cong \mathbf{Z}$.*

PROOF. By suspending appropriately, we may assume that $M \in E(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \mathbf{F}_2$. We may also assume that M is reduced, i.e., $Q_1 Q_0 = 0$. Let $0 \neq \langle [x] \rangle = H(M, Q_1)$, so that $Q_1(x) = 0$ and $x \notin \mathrm{Im}(Q_1)$. There are two possibilities:

(1) $Sq^1 x \neq 0$
(2) $Sq^1 x = 0$

In the first case, $Q_1 Sq^1 x = 0$ because M is reduced, so $Sq^1 x = Q_1 x_1$ for some x_1. (This because $[x]$ is the only nonzero Q_1 homology class of M.) Again, M reduced implies that $x_1 \notin \mathrm{Im}(Sq^1)$, so that $Sq^1 x_1 \neq 0$. Since $Q_1 Sq^1 x_1 = 0$, we have $Sq^1 x_1 = Q_1 x_2$ for some x_2. Continuing in this way, it follows by induction that M is not bounded-below, contrary to our assumption.

Therefore, we must have $Sq^1 x = 0$. Then $x = Sq^1 x_0$ for some x_0 and $x_0 \notin \mathrm{Im}(Q_1)$ because M is reduced. Hence $Q_1 x_0 \neq 0$. Again, the fact that M is reduced implies that $Q_1 x_0 = Sq^1 x_1$ for some x_1. For induction, we may suppose that we have found a sequence of elements x_i such that $Q_1 x_{i-1} = Sq^1 x_i \neq 0$, for $0 \leq i \leq n$. Then, since M is reduced, $x_n \notin \mathrm{Im}(Q_1)$, so $Q_1 x_n \neq 0$ and there must be x_{n+1} such that $Q_1 x_{n+1} = Sq^1 x_n$.

The submodule of M generated by the x_i is isomorphic to P_0 and the inclusion $P_0 \longrightarrow M$ induces an isomorphism of Q_k homologies, hence is a stable isomorphism by Theorem 2.6.

Now suppose that $M \in \mathrm{Pic}^{(1)}(E(1))$. By Lemma 6.5, $H(M, Q_1) = \Sigma^s \mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^s P_0$. Finally, observe that the $\Sigma^i P_0$ are all distinct because $H(\Sigma^i P_0, Q_1) = \Sigma^i \mathbf{F}_2$. □

Here is the result for $\mathcal{A}(1)$.

THEOREM 6.9 ([**16**, Theorem 2.1]). *If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \Sigma^a \mathbf{F}_2$ then $M \simeq \Sigma^{a-2b} P_b$ for some b. Therefore, $\mathrm{Pic}^{(1)}(\mathcal{A}(1)) = \{\Sigma^i P_n\} \cong \mathbf{Z} \oplus \mathbf{Z}/(4)$ with $(a, b) \in \mathbf{Z} \oplus \mathbf{Z}/(4)$ corresponding to $\Sigma^{a-2b} P_b$.*

PROOF. The first statement is the key technical result, and will be given as Theorem 7.1 in the next section. For the remainder, suppose that $M \in \mathrm{Pic}^{(1)}(\mathcal{A}(1))$. By Lemma 6.5, the first statement applies to show that $M = \Sigma^i P_n$ for some i and n. The multiplicative structure then follows from Theorem 4.5. □

7. The proof of Yu's Theorem

THEOREM 7.1. *If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \Sigma^a \mathbf{F}_2$ then $M \simeq \Sigma^{a-2b} P_b$ for some b.*

PROOF. We will assume that $a = 0$. We may also assume that M is reduced: $Sq^2 Sq^2 Sq^2$ acts as 0 on M. By Theorem 6.8, as an $E(1)$-module we have

$$M|_{E(1)} \cong P_0 \oplus (E(1) \otimes V)$$

for some bounded-below graded $\mathbf{F}_2$-vector space V. Recall that $\mathcal{A}(1)$ is generated by $E(1)$ and Sq^2. Therefore, to describe M as an $\mathcal{A}(1)$-module, it remains to specify the action of Sq^2 on M in a manner consistent with its structure as an $E(1)$-module. This is given by the following cocycle data. First, we have

(1) a linear functional $s : V \longrightarrow \mathbf{F}_2$, and
(2) linear transformations
 (a) $u : V_i \longrightarrow V_{i+1}$,
 (b) $v : V_i \longrightarrow V_{i-1}$, and
 (c) $w : V_i \longrightarrow V_{i-2}$,

such that, for all $y \in V$,

$$Sq^2 y = s(y)x_{t(y)} + Sq^1 u(y) + Q_1 v(y) + Sq^1 Q_1 w(y). \tag{7.1}$$

Here, $t(y) = 2 + |y|$ and x_t is the nonzero element of P_0 in degree t when $t \geq -1$. If $t < -1$ we take x_t to be 0, though we will see shortly that this cannot occur. There can be no term in V itself since M is reduced.

Similarly, we have sequences indexed on the integers $i \geq -1$:

(1) $a_i \in \mathbf{F}_2$,
(2) $b_i \in V_{i+1}$,
(3) $c_i \in V_{i-1}$, and
(4) $d_i \in V_{i-2}$,

such that

$$Sq^2 x_i = a_i x_{i+2} + Sq^1 b_i + Q_1 c_i + Sq^1 Q_1 d_i. \tag{7.2}$$

Again, there can be no term in V itself since M is reduced. It will be convenient to declare a_i, b_i, c_i and d_i to be 0 when $i < -1$.

Our main tools will be the direct sum decomposition (over $\mathbf{F}_2$)

$$M \;\cong\; P_0 \oplus V \oplus Sq^1 V \oplus Q_1 V \oplus Sq^1 Q_1 V$$

and the observation that the elements of $E(1)$ act monomorphically on V.

We now need a series of Lemmas.

First, consider the consequences of the relation $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$ on P_0.

LEMMA 7.2. *The action of Sq^2 on P_0 satisfies the following relations:*

(1) $a_{2i-1} + a_{2i} = 1$ *for* $i \geq 0$,
(2) $b_{2i} = 0$,
(3) $c_{2i} = 0$, *and*
(4) $d_{2i} = c_{2i-1}$.

PROOF. From equation (7.2) we have

$$\begin{aligned} Sq^1 Sq^2 x_i &= i a_i x_{i+3} + Sq^1 Q_1 c_i \\ Sq^2 Sq^1 x_i &= i(a_{i+1} x_{i+3} + Sq^1 b_{i+1} + Q_1 c_{i+1} + Sq^1 Q_1 d_{i+1}) \end{aligned}$$

Since $Q_1 x_i = i x_{i+3}$, comparing coefficients of the direct sum decomposition of M gives

$$i(1 + a_i + a_{i+1}) = 0$$

for $i \geq -1$, together with

$$\begin{aligned} i b_{i+1} &= 0 \\ i c_{i+1} &= 0 \\ c_i + i d_{i+1} &= 0 \end{aligned}$$

for all i. This implies the relations given. □

Next, we consider the action of Sq^2 on the free $E(1)$-module generated by V.

LEMMA 7.3. *Let $y \in V$ and $t = t(y)$. Then*
(1) $Sq^1 Sq^2 y = s(y)t(y)x_{t+1} + Sq^1 Q_1 v(y)$,
(2) $Sq^2 Sq^1 y = Q_1 y + s(y)t(y)x_{t+1} + Sq^1 Q_1 v(y)$, *and*
(3) $Sq^2 Q_1 y = s(y)t(y) \left(a_{t+1}x_{t+3} + Sq^1 b_{t+1} + Q_1 c_{t+1} + Sq^1 Q_1 d_{t+1}\right)$.

PROOF. Applying Sq^1 to equation (7.1) gives

$$Sq^1 Sq^2 y = s(y)t(y)x_{t+1} + Sq^1 Q_1 v(y).$$

We must then have

$$\begin{aligned} Sq^2 Sq^1 y &= Q_1 y + Sq^1 Sq^2 y \\ &= Q_1 y + s(y)t(y)x_{t+1} + Sq^1 Q_1 v(y) \end{aligned}$$

and

$$\begin{aligned} Sq^2 Q_1 y = Sq^2 Sq^1 Sq^2 y &= Sq^2 \left(s(y)t(y)x_{t+1} + Sq^1 Q_1 v(y)\right) \\ &= s(y)t(y) Sq^2 x_{t+1} \\ &= s(y)t(y) \left(a_{t+1}x_{t+3} + Sq^1 b_{t+1} + Q_1 c_{t+1} + Sq^1 Q_1 d_{t+1}\right) \end{aligned}$$

where we have used that $Sq^2 Sq^2 Sq^1 = 0$ and that $Sq^2 Sq^1 Q_1$ acts trivially since M is reduced. □

Now we turn to the consequences of the relation $Sq^2 Sq^2 = Sq^1 Q_1$ on V. These give stringent restrictions on the vector space V.

LEMMA 7.4. *Each V_i is at most one-dimensional. In addition, we have the following.*
(1) $V_i = 0$ *if* $i < -3$.
(2) V_{2i-2} *is spanned by* d_{2i}. *If it is nonzero, then* $a_{2i} = 0$, *and*

$$s(d_{2i}) + s(v(d_{2i})) = 1.$$

(3) V_{2i-1} *is spanned by* $d_{2i+1} + v(c_{2i+1})$. *If it is nonzero, then* $a_{2i+1} = 0$, $b_{2i+1} = 0$,

$$s(d_{2i+1} + v(c_{2i+1})) = 1,$$

and

$$c_{2i+1} = u(d_{2i+1} + v(c_{2i+1})).$$

PROOF. Applying the preceding Lemma to $y \in V_{t-2}$, we find

$$\begin{aligned} Sq^1 Q_1 y = Sq^2 Sq^2 y &= Sq^2 \left(s(y)x_t + Sq^1 u(y) + Q_1 v(y) + Sq^1 Q_1 w(y)\right) \\ &= s(y) \left(a_t x_{t+2} + Sq^1 b_t + Q_1 c_t + Sq^1 Q_1 d_t\right) \\ &\quad + Q_1 u(y) + s(u(y))(1+t)x_{t+2} + Sq^1 Q_1 v(u(y)) \\ &\quad + s(v(y))(1+t) \left(a_t x_{t+2} + Sq^1 b_t + Q_1 c_t + Sq^1 Q_1 d_t\right). \end{aligned}$$

where we let t = t(y). Separating terms from distinct summands, we get

$$\begin{aligned} 0 &= s(y)a_t + s(u(y))(1+t) + s(v(y))(1+t)a_t \\ 0 &= \left(s(y) + s(v(y))(1+t)\right) b_t \\ 0 &= \left(s(y) + s(v(y))(1+t)\right) c_t + u(y) \\ y &= \left(s(y) + s(v(y))(1+t)\right) d_t + v(u(y)) \end{aligned}$$

If $t = 2i$, then putting $c_{2i} = 0$ in the third equation implies that $u : V_{2i-2} \longrightarrow V_{2i-1}$ is 0. Then $v(u(y)) = 0$ as well, so that the last equation gives

$$y = (s(y) + s(v(y)))\, d_{2i}.$$

Hence V_{2i-2} is at most one dimensional, spanned by d_{2i}. If $d_{2i} \neq 0$ then, letting $y = d_{2i}$ in this equation gives

$$s(d_{2i}) + s(v(d_{2i})) = 1.$$

The first of our 4 summands then gives

$$\begin{aligned} 0 &= (s(d_{2i}) + s(v(d_{2i})))\, a_{2i} + s(u(d_{2i})) \\ &= a_{2i} + s(0) \\ &= a_{2i}. \end{aligned}$$

In the other parity, $t = 2i + 1$, so that $|y| = 2i - 1$, we get

$$\begin{aligned} 0 &= s(y)a_{2i+1} \\ 0 &= s(y)b_{2i+1} \\ u(y) &= s(y)c_{2i+1} \\ y &= s(y)d_{2i+1} + v(u(y)) \\ &= s(y)\left(d_{2i+1} + v(c_{2i+1})\right). \end{aligned}$$

We again find that V_{2i-1} is at most one dimensional, spanned now by $d_{2i+1} + v(c_{2i+1})$. If this is non-zero, then letting $y = d_{2i+1} + v(c_{2i+1})$ in the last equation gives $s(y) = 1$, from which it follows that $a_{2i+1} = 0$, $b_{2i+1} = 0$, and $c_{2i+1} = u(d_{2i+1} + v(c_{2i+1}))$.

Since the lowest degree nonzero d_i is d_{-1}, this gives $V_i = 0$ for $i < -3$. □

The action of $Sq^2Sq^2 = Sq^1Q_1$ on P_0 is already determined by the $E(1)$-module structure of M. This eliminates most of the possibilities left open by the preceding Lemma. We handle the even and odd degree cases separately because their proofs are somewhat different.

Lemma 7.5. *If $V_{2i-2} \neq 0$ then $2i - 2 = -2$.*

Proof. Lemma 7.4 implies that if $V_{2i-2} \neq 0$ then $V_{2i-2} = \langle d_{2i} \rangle$ and $a_{2i} = 0$. Lemma 7.2 then gives $a_{2i-1} = 1$, and we have

$$Sq^2x_{2i-1} = x_{2i+1} + Sq^1b_{2i-1} + Q_1d_{2i} + Sq^1Q_1d_{2i-1}$$

and

$$Sq^2x_{2i} = Sq^1Q_1d_{2i}.$$

If $2i - 2 \neq -2$ then $2i - 2 \geq 0$ by Lemma 7.4, and thus $2i - 3 \geq -1$. Then we have

$$\begin{aligned} 0 &= Sq^2Sq^2x_{2i-3} \\ &= Sq^2(a_{2i-3}x_{2i-1} + Sq^1b_{2i-3} + Q_1c_{2i-3} + Sq^1Q_1d_{2i-3}) \\ &= a_{2i-3}\left(x_{2i+1} + Sq^1b_{2i-1} + Q_1d_{2i} + Sq^1Q_1d_{2i-1}\right) \\ &\quad + Q_1b_{2i-3} + Sq^1Q_1v(b_{2i-3}) \\ &\quad + 0 \\ &= a_{2i-3}x_{2i+1} + Sq^1b_{2i-1} + Q_1(b_{2i-3} + d_{2i}) + Sq^1Q_1(b_{2i-3} + d_{2i}). \end{aligned}$$

Hence, $a_{2i-3} = 0$, so by Lemma 7.2, $a_{2i-2} = 1$. We therefore have

$$\begin{aligned} 0 &= Sq^2 Sq^2 x_{2i-2} \\ &= Sq^2(x_{2i} + Sq^1 Q_1 d_{2i-2}) \\ &= Sq^1 Q_1 d_{2i} \end{aligned}$$

which is a contradiction. □

LEMMA 7.6. *If $V_{2i-1} \neq 0$ then $2i-1 = -3$.*

PROOF. Lemma 7.4 implies that if $V_{2i-1} \neq 0$ and $2i-1 \neq -3$ then $2i-1 \geq -1$. Also, $d_{2i+1} + v(c_{2i+1}) \neq 0$, $a_{2i+1} = 0$ and $b_{2i+1} = 0$. By Lemma 7.2, it follows that $a_{2i+2} = 1$. Then

$$Sq^2 x_{2i+1} = Q_1 c_{2i+1} + Sq^1 Q_1 d_{2i+1}$$

and

$$Sq^2 x_{2i+2} = x_{2i+4} + Sq^1 Q_1 c_{2i+1}.$$

(Recall from Lemma 7.2 that $c_{2i+1} = d_{2i+2}$.) Then,

$$\begin{aligned} 0 &= Sq^2 Sq^2 x_{2i} \\ &= Sq^2(a_{2i} x_{2i+2} + Sq^1 Q_1 d_{2i}) \\ &= a_{2i}\left(x_{2i+4} + Sq^1 Q_1 c_{2i+1}\right) \end{aligned}$$

Hence, $a_{2i} = 0$, so by Lemma 7.2, $a_{2i-1} = 1$. We therefore have

$$\begin{aligned} 0 &= Sq^2 Sq^2 x_{2i-1} \\ &= Sq^2(x_{2i+1} + Sq^1 b_{2i-1} + Q_1 c_{2i-1} + Sq^1 Q_1 d_{2i-1}) \\ &= Q_1 c_{2i+1} + Sq^1 Q_1 d_{2i+1} \\ &\quad + Q_1 b_{2i-1} + Sq^1 Q_1 v(b_{2i-1}) \\ &\quad + 0. \end{aligned}$$

The Q_1 component implies that $b_{2i-1} = c_{2i+1}$. But then, the $Sq^1 Q_1$ component is $Sq^1 Q_1(d_{2i+1} + v(c_{2i+1})) \neq 0$, which is a contradiction. □

PROOF OF 7.1 CONTINUED. Now we can finish the proof. Certainly V_{-3} and V_{-2} cannot both be nonzero, since the first implies $a_{-1} = 0$ and the second implies $a_0 = 0$, but we must have $a_{-1} + a_0 = 1$ by Lemma 7.2.

If both are 0, then $M|_{E(1)} \cong P_0$. Lemma 7.2 gives $a_{2i-1} + a_{2i} = 1$, while $0 = Sq^1 Q_1 x_i = Sq^2 Sq^2 x_i$ gives $a_i a_{i+2} = 0$. The entire $\mathcal{A}(1)$ action is thus determined by a_{-1}. It follows that $M \cong P_0$ or $M \cong \Sigma^{-2} P_1$.

If $V_{-3} \neq 0$, then $y = d_{-1} \neq 0$, while $c_{-1} = 0$ since $V_{-2} = 0$. Also, $a_{-1} = 0$ and $s(y) = 1$. Therefore, Lemma 7.3 gives

$$\begin{aligned} Sq^2 y &= x_{-1} \\ Sq^2 Sq^1 y &= Q_1 y + x_0 \\ Sq^2 Q_1 y &= x_2. \end{aligned}$$

With the exception of $Sq^2 x_{-1} = Sq^2 Sq^2 y = Sq^1 Q_1 y$, the action of Sq^2 on the x_i alternates as in the case $V = 0$. It follows that $M \cong \Sigma^{-6} P_3$ under the isomorphism which takes y to the bottom class, 111, and x_1 to the indecomposable in degree 1, $124 + 142 + 421$, in the notation of Section A. (See Figure 2).

Finally, if $V_{-2} \neq 0$, then $V_{-2} = \langle d_0 \rangle$, $a_0 = 0$, $a_{-1} = 1$, and the b_i, c_i and d_i are all 0 except for $c_{-1} = d_0$. We get

$$\begin{aligned} Sq^2 d_0 &= x_0 \\ Sq^2 Sq^1 d_0 &= Q_1 d_0 \\ Sq^2 Q_1 d_0 &= 0, \end{aligned}$$

while

$$\begin{aligned} Sq^2 x_{-1} &= x_1 + Q_1 d_0 \\ Sq^2 x_0 &= Sq^1 Q_1 d_0. \end{aligned}$$

The remaining $Sq^2 x_i$ are as in P_0. This is isomorphic to $\Sigma^{-4} P_2$ by the isomorphism under which d_0 generates the Joker, while R is the submodule spanned by

$$x_{-1} + Sq^1 d_0,\ x_0,\ x_1 + Q_1 d_0,\ x_2 + Sq^1 Q_1 d_0,\ x_3,\ x_4, \ldots \qquad \square$$

8. $\mathrm{Pic}^{(k)}$ continued

We now turn to the determination of the groups $\mathrm{Pic}^{(0)}$. For $E(1)$, the argument is similar to that for $\mathrm{Pic}^{(1)}$.

PROPOSITION 8.1. *If* $M \in E(1)\text{-Mod}^{(0)}$ *and* $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ *then* $M \simeq \Sigma^{s+1} R$. *Therefore,* $\mathrm{Pic}^{(0)}(E(1)) = \{\Sigma^i R\} \cong \mathbf{Z}$.

PROOF. By suspending appropriately, we may assume that $M \in E(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^{-1} \mathbf{F}_2$. We may also assume that M is reduced.

Let $0 \neq \langle [x] \rangle = H(M, Q_0)$, so that $Sq^1 x = 0$ and $x \notin \mathrm{Im}(Sq^1)$. There are two possibilities:

(1) $Q_1 x = 0$
(2) $Q_1 x \neq 0$

In the first case, $x = Q_1 y_0$ for some y_0, which cannot be in the image of Sq^1, since M is reduced, so that $Sq^1 y_0 \neq 0$. We may then assume for induction that we are given y_i such that $Q_1 y_i = Sq^1 y_{i-1}$ for $0 \leq i \leq n$, and such that $Q_1 y_0 = x$ and $Sq^1 y_n \neq 0$. The assumption that M is reduced allows us to extend this another step, completing the induction. We conclude that M is not bounded-below, contrary to assumption.

It therefore follows that $Q_1 x \neq 0$. Then $Sq^1 Q_1 x = 0$ because M is reduced, so $Q_1 x = Sq^1 x_1$ for some x_1. Again, M reduced implies that $Q_1 x_1 \neq 0$. We may assume for induction that we have elements x_i with $Sq^1 x_i = Q_1 x_{i-1} \neq 0$ for $0 \leq i \leq n$ and $Q_1 x_n \neq 0$. (Let $x_0 = x$ here.) Then M reduced implies $Q_1 x_n = Sq^1 x_{n+1}$ for some x_{n+1} and $Q_1 x_{n+1} \neq 0$, completing the induction. The x_i generate a submodule isomorphic to R and the inclusion $R \longrightarrow M$ induces a stable isomorphism.

Now suppose that $M \in \mathrm{Pic}^{(0)}(E(1))$. By Lemma 6.5, $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^{s+1} R$. Finally, observe that the $\Sigma^i R$ are all distinct because $H(\Sigma^i R, Q_0) = \Sigma^{i-1} \mathbf{F}_2$. $\square$

For $\mathcal{A}(1)$, the argument is a bit more complicated, but the conclusion is the same.

PROPOSITION 8.2. *If* $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ *and* $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ *then* $M \simeq \Sigma^{s+1} R$. *Therefore,* $\mathrm{Pic}^{(0)}(\mathcal{A}(1)) = \{\Sigma^i R\} \cong \mathbf{Z}$.

Proof. By suspending appropriately, we may assume that $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^{-1}\mathbf{F}_2$. We may also assume that M is reduced: $Sq^2Sq^2Sq^2$ acts as 0 on M. Let $0 \neq \langle [x] \rangle = H(M, Q_0)$, so that $Sq^1x = 0$ and $x \notin \text{Im}(Sq^1)$.

Let $M \cong M_0 \oplus M_1$ as an $E(1)$-module, where M_0 is a reduced $E(1)$-module and M_1 is $E(1)$-free. Then M_0 is in $\text{Pic}^{(0)}(E(1))$ with $H(M_0, Q_0) = H(M, Q_0) = \langle [x] \rangle$. By the preceding Proposition, $M_0 \cong R$. We may choose $x \in M_0$. In particular, $Q_1x \neq 0$. Since $Sq^1x = 0$, $Q_1x \neq 0$ implies that $Sq^1Sq^2x \neq 0$. There are two possibilities:

(1) $Sq^2Sq^1Sq^2x \neq 0$
(2) $Sq^2Sq^1Sq^2x = 0$

In the first case, the submodule $\langle x \rangle$ is $\Sigma^{-1}\mathcal{A}(1)/\!\!/\mathcal{A}(0)$ since M is reduced and $Sq^1x = 0$. The long exact sequences in Q_k-homology induced by the short exact sequence

$$0 \longrightarrow \langle x \rangle \longrightarrow M \longrightarrow M/\langle x \rangle \longrightarrow 0$$

imply that $H(M/\langle x \rangle, Q_1) = 0$ and $H(M/\langle x \rangle, Q_0) = \langle [y] \rangle$ with $Q_0y = Sq^2Sq^1Sq^2x$. Then $M/\langle x \rangle$ satisfies the same hypotheses as M shifted up by 4 degrees. We can thus inductively construct $R \longrightarrow M$ inducing an isomorphism in Q_0 and Q_1 homology. Hence M is stably isomorphic to R as claimed.

The second alternative implies that the submodule generated by x is spanned by x, Sq^2x and Sq^1Sq^2x. This has Q_1 homology $\langle [Sq^2x] \rangle$. The long exact homology sequence for

$$0 \longrightarrow \langle x \rangle \longrightarrow M \longrightarrow M/\langle x \rangle \longrightarrow 0$$

then implies that $H(M/\langle x \rangle, Q_0) = 0$ and $H(M/\langle x \rangle, Q_1) = \langle [y] \rangle$ with $Q_1y = Sq^2x$. By Yu's theorem (Theorem 6.9), $M/\langle x \rangle$ must be a suspension of P_n for some n. (It is actually isomorphic to Σ^iP_n, not just stably equivalent to it, because it is reduced, being a quotient of the reduced module M.) Further, if we let $y \in M$ be a class whose image in $M/\langle x \rangle$ generates $H(M/\langle x \rangle, Q_1)$ then $Q_1y = Sq^2x$. Now $Sq^1y = 0$ because this is so in each P_n and because x, which is in the same degree as Sq^1y, is not in the image of Sq^1. Thus, we must have $Sq^1Sq^2y = Sq^2x$. This is impossible. In P_0, $Sq^2y = 0$, while in P_n, $1 \leq n \leq 3$, Sq^2y is in the image of Sq^1. Since $\langle x \rangle$ is zero in this degree, the same holds in M. This contradiction shows that the second alternative does not happen, proving the theorem.

Now suppose that $M \in \text{Pic}^{(0)}(\mathcal{A}(1))$. By Lemma 6.5, $H(M, Q_0) = \Sigma^s\mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^{s+1}R$. Finally, observe that the Σ^iR are all distinct because $H(\Sigma^iR, Q_0) = \Sigma^{i-1}\mathbf{F}_2$. □

From these last four results, we have the converse of Lemma 6.5.

Corollary 8.3. *A module $M \in B\text{-Mod}^{(k)}$ is in $\text{Pic}^{(k)}(B)$ iff $H(M, Q_k)$ is one dimensional.*

It is useful to have explicit forms for these isomorphisms between the torsion-free quotient of $\text{Pic}^{(k)}(B)$ and $\mathbf{Z}$.

Corollary 8.4. *For $M \in \text{Pic}^{(k)}(B)$, let $d_k(M)$ be defined by $H(M, Q_k) = \Sigma^{d_k(M)}\mathbf{F}_2$. Then $d_k : \text{Pic}^{(k)}(B) \longrightarrow \mathbf{Z}$ is a homomorphism. It is an isomorphism if $k = 0$ or $B = E(1)$. When $k = 1$ and $B = \mathcal{A}(1)$, $\text{Ker}(d_1) = \{\Sigma^{-2i}P_i\} \cong \mathbf{Z}/(4)$.*

Proof. The Künneth isomorphism implies that d_k is a homomorphism. The remainder follows directly from Theorems 6.9 and 6.8, and Propositions 8.1 and 8.2. ☐

When $k = 1$ and $B = \mathcal{A}(1)$ we need another invariant to detect $\mathrm{Ker}(d_1)$. It is possible to define it directly in terms of M by considering divisibility of elements in $\mathrm{Ext}^1_{\mathcal{A}(1)}(M, \mathbf{F}_2)$, but this is cumbersome to define, so we content ourselves with an invariant defined in terms of M^{red}.

Proposition 8.5. *If $M \in \mathrm{Pic}^{(1)}(\mathcal{A}(1))$, let*

- *c be the connectivity (bottom non-zero degree) of M^{red},*
- *$e = \dim(Sq^2(M^{\mathrm{red}}_c))$, and*
- *$f = \dim(Sq^2 Sq^2(M^{\mathrm{red}}_c))$.*

(Here $\dim$ refers to dimension as an $\mathbf{F}_2$ vector space.) Let $t_1(M) = d_1(M) - c - e + f$. Then $t_1 : \mathrm{Pic}^{(1)}(\mathcal{A}(1)) \longrightarrow \mathbf{Z}/(4)$ is a homomorphism and $M \simeq \Sigma^{d_1(M) - 2t_1(M)} P_{t_1(M)}$.

Proof. It is simplest to reverse engineer this. We compute these invariants for $\Sigma^i P_n$:

	$\Sigma^i P_0$	$\Sigma^i P_1$	$\Sigma^i P_2$	$\Sigma^i P_3$
c	i-1	i+1	i+2	i+3
d_1	i	i+2	i+4	i+6
e	1	0	1	1
f	0	0	1	1
$t_1 = d_1 - c - e + f$	0	1	2	3

Theorem 4.5 shows that t_1 is a homomorphism. The equivalence between M and $\Sigma^{d_1(M) - 2t_1(M)} P_{t_1(M)}$ is evident from the table above. ☐

9. The homomorphisms from Pic to $\mathrm{Pic}^{(k)}$

Over a finite dimensional graded Hopf algebra, the Picard group always contains suspension and loops. This accounts for the $\mathbf{Z} \oplus \mathbf{Z}$ found by Adams and Priddy (Theorems 6.6 and 6.7) in $\mathrm{Pic}(E(1))$ and $\mathrm{Pic}(\mathcal{A}(1))$. In the Picard groups of the localized subcategories B-$\mathrm{Mod}^{(k)}$ these become dependent: $\Sigma(\mathbf{L}_0\mathbf{F}_2) = \Omega(\mathbf{L}_0\mathbf{F}_2)$ and $\Sigma^3(\mathbf{L}_1\mathbf{F}_2) = \Omega(\mathbf{L}_1\mathbf{F}_2)$ over $E(1)$, for example.

Together, however, the functors $\mathbf{L}_i$ give an embedding of Pic into the localized Picard groups.

Proposition 9.1. *Each $\mathbf{L}_k : \mathrm{Pic}(E(1)) \longrightarrow \mathrm{Pic}^{(k)}(E(1))$ is an epimorphism. Their product $\mathbf{L}$, mapping $\mathrm{Pic}(E(1))$ to $\mathrm{Pic}^{(0)}(E(1)) \oplus \mathrm{Pic}^{(1)}(E(1))$, is a monomorphism with cokernel $\mathbf{Z}/(2)$. With respect to the basis $\{\Sigma\mathbf{F}_2, \Omega\mathbf{F}_2\}$ of Pic we have*

$$\begin{array}{ccc}
\mathrm{Pic}(E(1)) & \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}} & \\
\Big\downarrow \mathbf{L} & & \searrow \\
\mathrm{Pic}^{(0)}(E(1)) \oplus \mathrm{Pic}^{(1)}(E(1)) & \xrightarrow[d_0 \oplus d_1]{} & \mathbf{Z} \oplus \mathbf{Z}
\end{array}$$

PROOF. Explicitly, $\mathbf{L}(M) = (\mathbf{L}_0 M, \mathbf{L}_1 M) = (\Sigma R \otimes M, P_0 \otimes M)$. We simply compute:

$$d_0(\Sigma R \otimes \Sigma \mathbf{F}_2) = d_0(\Sigma^2 R) = 1$$

and

$$d_1(P_0 \otimes \Sigma \mathbf{F}_2) = d_1(\Sigma P_0) = 1,$$

while

$$d_0(\Sigma R \otimes \Omega \mathbf{F}_2) = d_0(\Omega \Sigma R) = d_0(\Sigma^2 R) = 1$$

and

$$d_1(P_0 \otimes \Omega \mathbf{F}_2) = d_1(\Omega P_0) = d_1(\Sigma^3 P_0) = 3.$$ □

Over $\mathcal{A}(1)$ we also have the torsion summands to consider.

PROPOSITION 9.2. *The restriction maps*

$$\mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}(E(1))$$

and

$$\mathrm{Pic}^{(k)}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(k)}(E(1))$$

induce isomorphisms from the torsion free quotients of their domains to their codomains, and commute with $\mathbf{L}$. *Each* $\mathbf{L}_k : \mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(k)}(\mathcal{A}(1))$ *is an epimorphism. With respect to the basis* $\{\Sigma \mathbf{F}_2, \Omega \mathbf{F}_2, J\}$ *of* Pic, *the homomorphism* $\mathbf{L} : \mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(0)}(\mathcal{A}(1)) \oplus \mathrm{Pic}^{(1)}(\mathcal{A}(1))$ *is*

$$\begin{array}{ccc}
\mathrm{Pic}(\mathcal{A}(1)) & \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & \overline{1} & \overline{2} \end{bmatrix}} & \\
\Big\downarrow \mathbf{L} & & \searrow \\
\mathrm{Pic}^{(0)}(\mathcal{A}(1)) \oplus \mathrm{Pic}^{(1)}(\mathcal{A}(1)) & \xrightarrow[\begin{bmatrix} d_0 & 0 \\ 0 & d_1 \\ 0 & t_1 \end{bmatrix}]{} & \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/(4)
\end{array}$$

with $\overline{k}$ *denoting the coset* $k + (4)$. *The cokernel of* $\mathbf{L}$ *is* $\mathbf{Z}/(4)$.

PROOF. Again, we simply compute. The d_0 and d_1 calculations are the same as for $E(1)$. This implies the first claim and gives the upper left two by two submatrix. For the remainder, we first compute $\mathbf{L}_1$. We have $\mathbf{L}_1(\Sigma \mathbf{F}_2) = \Sigma P_0$, which projects to $\overline{0}$ in the $\mathbf{Z}/(4)$ summand. We also have $\mathbf{L}_1(\Omega \mathbf{F}_2) = \Omega P_0 = \Sigma^1 P_1$, which projects to $\overline{1}$ in the $\mathbf{Z}/(4)$ summand. Next,

$$d_0(\mathbf{L}_0(J)) = d_0(\Sigma R \otimes J) = 0$$

$$d_1(\mathbf{L}_1(J)) = d_1(P_0 \otimes J) = 0.$$

Finally, $P_0 \otimes J$ is stably isomorphic to $\Sigma^{-4} P_2$. This follows by tensoring the short exact sequence containing $M_2 = \Sigma^4 J$ of Theorem 4.6 with P_0. Since $P_0 \otimes R$ is free by Theorem 2.6, this gives an equivalence $P_0 \otimes J = P_0 \otimes \Sigma^{-4} M_2 \simeq P_0 \otimes \Sigma^{-4} P_2 \simeq \Sigma^{-4} P_2$.

Determination of the cokernel is a simple Smith Normal Form calculation. □

10. Idempotents and localizations

Again let B be either $E(1)$ or $\mathcal{A}(1)$. In this section we show that $\mathbf{L}_0$ and $\mathbf{L}_1$ are essentially unique, in that the only stably idempotent modules in B-Mod^b are ones we have already seen.

THEOREM 10.1. *If $M \in B\text{-}\mathrm{Mod}^b$ is stably idempotent then M is stably equivalent to one of 0, $\mathbf{F}_2$, P_0, ΣR, or $P_0 \oplus \Sigma R$.*

PROOF. We give the proof for $B = \mathcal{A}(1)$. The proof for $E(1)$ is similar but easier.

We first note a simple fact: if $M \otimes M \simeq M$ then each $H(M, Q_i)$ must be either 0 or $\mathbf{F}_2$. This yields four possibilities.

If both are 0, then $0 \longrightarrow M$ is a stable equivalence by Theorem 2.6.

If exactly one Q_i-homology group is nonzero, we have the unit in $\mathrm{Pic}^{(i)}(\mathcal{A}(1))$, which must be either P_0 or ΣR by Theorems 6.9 and 8.2.

The final possibility is that $H(M, Q_0) = \mathbf{F}_2 = H(M, Q_1)$. In this case we tensor M with the triangle

$$\Sigma R \longrightarrow \mathbf{F}_2 \longrightarrow P_0$$

We get a triangle

$$\mathbf{L}_0 M \longrightarrow M \longrightarrow \mathbf{L}_1 M.$$

By Theorem 5.5, each $\mathbf{L}_i(M)$ is stably idempotent and Q_i-local. By the preceding paragraph, $\mathbf{L}_0(M) \simeq \Sigma R$ and $\mathbf{L}_1(M) \simeq P_0$. It remains to determine the possible extensions M.

It is a simple matter to verify that $\mathrm{Ext}^{1,0}_{\mathcal{A}(1)}(P_0, \Sigma R) = \mathbf{F}_2$. Therefore, the two possibilities are the split extension $M \simeq P_0 \oplus \Sigma R$ and the nonsplit $M \simeq \mathbf{F}_2$ above. □

11. A final example

As noted in 2.7, the detection of stable isomorphisms is more subtle in the category of all $\mathcal{A}(1)$-modules: the module $L = \mathbf{F}_2[x, x^{-1}]$ has trivial Q_0 and Q_1 homology, yet is not stably free. It provides another idempotent as well.

PROPOSITION 11.1. *As $\mathcal{A}(1)$-modules,* $L \otimes L \;\cong\; L \oplus \bigoplus_{i,j \in \mathbf{Z}} \Sigma^{4i+2j-2}\mathcal{A}(1).$

PROOF. The elements $x^{4i-1} \otimes x^{2j-1}$ generate a free submodule, and the submodule $\{x^i \otimes x^0 | i \in \mathbf{Z}\}$, which is isomorphic to L, is a complementary submodule. □

Therefore, we have another localization functor

$$\mathbf{L}_\infty(M) = L \otimes M.$$

The module L shows that Q_0 and Q_1 homology are insufficient to capture a more general notion of being Q_0 or Q_1 local.

PROPOSITION 11.2. $\mathbf{L}_0 L \;\simeq\; 0$ *and* $\eta_L : L \xrightarrow{\simeq} \mathbf{L}_1 L.$

PROOF. $\mathbf{L}_0 L = \Sigma R \otimes L$ is free over $\mathcal{A}(1)$ on the elements $x^{4i-1} \otimes x^{2j-1}$ with $i \geq 0$. The canonical triangle, $\mathbf{L}_0 L \longrightarrow L \xrightarrow{\eta_L} \mathbf{L}_1 L$ then shows that L is equivalent to its $\mathbf{L}_1$ localization. □

Appendix A. Locating P_n in $P^{\otimes(n)}$

The 'hit problem' is the problem of determining a set of $\mathcal{A}$-module generators of the polynomial rings $\mathbf{F}_2[x_1, \ldots, x_n] = H^*B(C_2^n)_+$. See [**4**] for a recent paper on the problem, and [**3**] for work on the problem using the results we prove here. One approach to it is to consider the analogous problem over subalgebras $\mathcal{A}(n)$. The results of section 4 simplify the problem in the case of $\mathcal{A}(1)$. Those results only identify the stable type, P_n, of $H^*(BC_2 \wedge \cdots \wedge BC_2)$. In this section we will produce explicit embeddings $P_n \longrightarrow P^{\otimes(n)}$. Naturally, there are choices involved, but the inductive determination of the isomorphism type also gives us a way to inductively find P_{n+1} as a summand of $P_n \otimes P \subset P^{\otimes(n)} \otimes P$, reducing the work dramatically.

Let us write $x_1^{i_1} \ldots x_n^{i_n}$ as $i_1 \ldots i_n$ and define $\overline{i_1 \ldots i_n}$ to be the orbit sum of $i_1 \ldots i_n$.

THEOREM A.1. *For $n > 0$, P_n can be embedded in $P^{\otimes(n)}$ as follows:*

- $M_1 = \langle 1, 2, 4 \rangle$
- $P_1 = P = M_1 + \langle 3, i \mid i \geq 5 \rangle$
- $M_2 = \langle 11, \overline{12}, 22, \overline{14}, \overline{24} \rangle$
- $P_2 = M_2 + \langle 21, 4i \mid i \geq 1 \rangle$
- $M_3 = \langle \overline{112}, 222 + \overline{114}, \overline{124} \rangle$
- $P_3 = M_3 + \langle 111, \overline{122}, 222, \overline{224}124 + 142 + 421, 44i \mid i \geq 1 \rangle$
- $M_4 = \langle 2222 + \overline{1124} \rangle$
- $P_4 = M_4 + \langle 2221 + \overline{1114}, \overline{1224} + \overline{1242}, 2224, \overline{224}i \mid i \geq 1 \rangle$
- $P_{n+4} \;\cong\; \langle 2222 + \overline{1124} \rangle \otimes P_n$.

REMARK A.2. There are several notable points about these submodules.

(1) The first three generators x_1, x_1x_2, and $x_1x_2x_3$, are obvious from the connectivity: the connectivity of P_n is n for $n < 4$.

(2) The fourth, $x_1^2x_2^2x_3^2x_4 + \overline{x_1x_2x_3x_4^4}$ in degree 7, is less so. The classes of degrees less than 7 all lie in free summands of $P_3 \otimes P$. Modulo those free summands, there are 4 possible choices for the degree 7 class in P_4:

$$2221 + \overline{1114} + \alpha_0(2221 + \overline{2212}) + \alpha_1(1114 + \overline{1123})$$

for $\alpha_i \in \{0, 1\}$.

(3) Applying Sq^1 to any of these four classes yields the same 'periodicity class' $B = 2222 + \overline{1124}$ in degree 8. From $H(P, Q_1) = [x_1^2]$, we know that $H(P_4, Q_1) = [x_1^2x_2^2x_3^2x_4^2]$, but since $Sq^2(x_1^2x_2^2x_3^2x_4^2) \neq 0$, the 'periodicity class' must have additional terms, which turn out to be exactly $Q_0Q_1(x_1x_2x_3x_4)$, or $\overline{1124}$ in our abbreviated notation.

(4) Above the bottom few degrees, each of the P_i can be written as the tensor product of an $\mathcal{A}(1)$-annihilated class with P. These $\mathcal{A}(1)$-annihilated classes are B^i, $x_1^4B^i$, $x_1^4x_2^4B^i$, and $\overline{x_1^2x_2^2x_3^4}B^i$.

PROOF. Evidently $P_1 = P$. For P_2, it is a simple matter to verify that x_1x_2 generates M_2. To finish P_2, clearly $x = x_1^2x_2$ serves, with the rest of P_2 then given by $x_1^4(x_2^i)$.

Expressing P_3 as the nontrivial extension of M_3 by $P_3/M_3 \;\cong\; \Sigma^4R$ requires that the bottom class of M_3 be $Sq^1(x_1x_2x_3) = \overline{112}$. The bottom $\mathcal{A}(1)/\!\!/\mathcal{A}(0)$ is forced, but for the second one, we need x with $Sq^1x = \overline{x_1^2x_2^2x_3^4}$. By choosing $x = x_1x_2^2x_3^4 + x_1x_2^4x_3^2 + x_1^4x_2^2x_3$, the rest of P_3 is given by $x_1^4x_2^4(x_3^i)$.

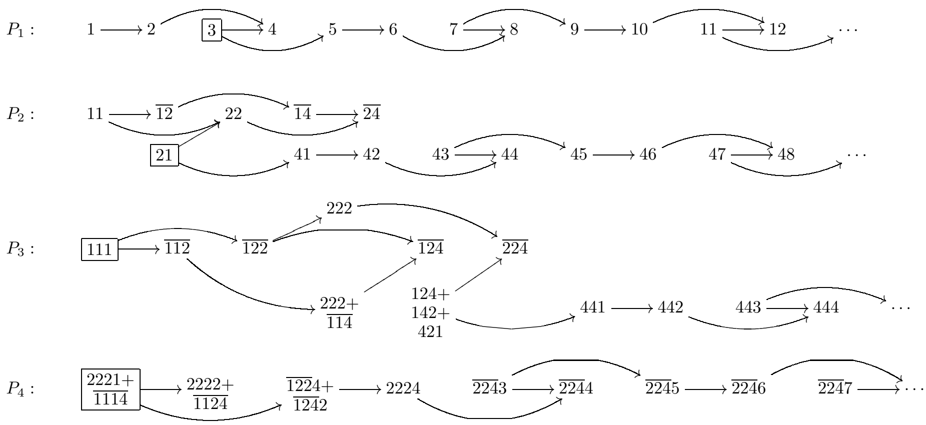


FIGURE 2. The modules P_n embedded in $P^{\otimes(n)}$, $1 \le n \le 4$. The bottom class of the quotient $\Sigma^t R$ is boxed. See Section A for notation.

For P_4, we need a class in degree 7 in $P_3 \otimes P$ which is not in $\operatorname{Im}(Sq^1) + \operatorname{Im}(Sq^2)$ and whose annihilator ideal is (Sq^2Sq^1). Solving $Sq^1x \ne 0$, $Sq^2Sq^1x = 0$, $Sq^2Sq^1Sq^2x \ne 0$, for $x \notin \operatorname{Im}(Sq^1) + \operatorname{Im}(Sq^2)$, we arrive at the 4 choices in Remark A.2.(2) above. Our choice, $\alpha_0 = \alpha_1 = 0$, gives the version of P_4/M_4 which is simplest to describe.

Finally, consider periodicity. Since M_4 is a trivial $\mathcal{A}(1)$ module, tensoring with it is the same as 8-fold suspension. Now, if we tensor the short exact sequence $0 \longrightarrow M_4 \longrightarrow P_4 \longrightarrow \Sigma^8 R \longrightarrow 0$ with P_n, we get

$$0 \longrightarrow M_4 \otimes P_n \longrightarrow P_4 \otimes P_n \longrightarrow \Sigma^8 R \otimes P_n \longrightarrow 0.$$

The Künneth theorem and Theorem 2.6 imply that $M_4 \otimes P_n$ is stably isomorphic to $P_4 \otimes P_n$, and hence to $P^{\otimes(4)} \otimes P^{\otimes(n)}$. Since $M_4 \otimes P_n$ is indecomposable, it follows that it is isomorphic to P_{n+4} and that the inclusion $M_4 \otimes P_n \subset P_4 \otimes P_n \subset P^{\otimes(4)} \otimes P^{\otimes(n)}$ serves our purpose. □

Appendix B. The free summand in $P^{\otimes(n)}$

We have now shown that if $n > 0$ then

$$P^{\otimes n} = P_n \oplus F_n$$

where F_n is a free $\mathcal{A}(1)$-module. We can therefore give a complete decomposition of $P^{\otimes(n)}$ by simply computing the Hilbert series of the free part. This can be found in Yu's thesis ([**16**, Theorem 4.2]). The most transparent form of the Hilbert series for the P_n can simply be read off from Theorem 4.6.

LEMMA B.1. *$H(P_{4k+i}) = t^{8k} H(P_i)$ and*

- $H(P_0) = \dfrac{t^{-1}}{1-t}$

- $H(P_1) = \dfrac{t^1}{1-t}$
- $H(P_2) = \dfrac{t^2}{1-t} + t^3 + t^5 + t^6$
- $H(P_3) = \dfrac{t^3}{1-t} + t^6 + t^7$

Another form works a bit better in connection with the Hilbert series for $P^{\otimes(n)}$.

LEMMA B.2. *The Hilbert series*

$$H(P_n) = \frac{t^{2n}}{1-t} Q_n(t)$$

where

$$Q_n(t) = \begin{cases} \dfrac{1}{t} & n \equiv 0,1 \pmod 4 \\ \dfrac{1+t-t^2+t^3-t^5}{t^2} & n \equiv 2 \pmod 4 \\ \dfrac{1+t^3-t^5}{t^3} & n \equiv 3 \pmod 4 \end{cases}$$

PROOF. Straightforward. □

We can now locate the summands in the free parts F_n.

THEOREM B.3. *The Hilbert series of the modules F_n are*

$$H(F_n) = H(\mathcal{A}(1)) \frac{t^n(1-t^n(1-t)^{n-1}Q_n(t))}{(1-t)^{n-1}(1-t^4)(1+t^3)}$$

PROOF. We simply compute

$$\begin{aligned} \frac{H(P^{\otimes n}) - H(P_n)}{H(\mathcal{A}(1))} &= \frac{\left(\dfrac{t}{1-t}\right)^n - \dfrac{t^{2n}}{1-t} Q_n}{(1+t)(1+t^2)(1+t^3)} \\ &= \frac{t^n - t^{2n}(1-t)^{n-1}Q_n(t)}{(1-t)^n(1+t)(1+t^2)(1+t^3)} \\ &= \frac{t^n(1-t^n(1-t)^{n-1}Q_n(t))}{(1-t)^{n-1}(1-t^4)(1+t^3)} \end{aligned}$$ □

The following special cases are of particular interest, and are the correct replacement for Lemma 2 in [**11**], where the free part of $P \otimes P$ is asserted to be $\mathcal{A}(1) \otimes \Sigma^2 \mathbf{F}_2[u_2, v_4]$.

COROLLARY B.4. *As $\mathcal{A}(1)$-modules*

$$P \otimes P_0 \cong P \oplus (\mathcal{A}(1) \otimes \mathbf{F}_2[u_2, v_4])$$

and

$$P \otimes P \cong P_2 \oplus \bigoplus_{\substack{i,j \geq 0 \\ i+j>0}} \Sigma^{4i+4j} \mathcal{A}(1) \oplus \bigoplus_{i,j \geq 0} \Sigma^{4i+4j+6} \mathcal{A}(1)$$

REMARK B.5. P_0 is the cohomology of $T(-\lambda)$, the Thom complex of the negative of the line bundle over $P = BC_2$. As a consequence, the first isomorphism in Corollary B.4 can be used to give a homotopy equivalence

$$ko \wedge BC_2 \wedge T(-\lambda) \simeq (ko \wedge BC_2) \vee H\mathbf{F}_2[u_2, v_4].$$

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