Commutative Ring Spectra and Spectral Sequences

Robert Bruner

Department of Mathematics
Wayne State University

Structured Ring Spectra - TNG
Hamburg, Germany
1–5 August 2011
Outline

1. Introduction
2. The Homotopy Fixed Point Spectral Sequence
3. The Adams Spectral Sequence
4. Related and future work
Attitudes toward spectral sequences

Three basic attitudes:
- dismissal
- okay when they collapse or map isomorphically
- hardcore

The hardcore contend with
- nontrivial differentials
- ‘hidden extensions’ or ‘filtration shifts’

My goal is to show how $S$-algebra or $E_{\infty}$ structures solve some of these problems.
Commutative $S$-algebras

The product factors through the homotopy orbits

\[
\begin{array}{c}
R^{(k)} \\
\downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
(R^{(k)})_{h\pi} \\
\end{array}
\]

\[
\begin{array}{c}
R^{(k)} \\
\downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \\
R \\
\end{array}
\]

\[
\begin{array}{c}
\mu \\
\xi \\
\end{array}
\]

for any subgroup $\pi \subset \Sigma_k$.

Traditional notation:

\[
D^i_\pi R := E^{i}_\pi \wedge_\pi R^{(k)}
\]

where $E^{i}_\pi$ is the $i$-skeleton of the universal $\pi$-space $E_\pi$. 

Let $p = 2$ and $H = H\mathbb{F}_p$ for most examples.
Let $C_2 = \Sigma_2 = \{1, T\}$ and consider the $\mathbb{F}_2[C_2]$ resolution of $\mathbb{F}_2$

$$
0 \leftarrow \mathbb{F}_2 \xleftarrow{\epsilon} \mathcal{W}_0 \leftarrow \mathcal{W}_1 \xleftarrow{d_1} \mathcal{W}_2 \xleftarrow{d_2} \ldots
$$

- $\mathcal{W}_i = \langle e_i \rangle \cong \mathbb{F}_2[C_2]$
- $d_i(e_{i+1}) = (1 + T)e_i$.
- $EC_2 = S(\infty, \tau)$ has cellular chains $\mathcal{W}$
- $H_* D_{C_2} R \cong H(\mathcal{W} \otimes_{C_2} H_*(R) \otimes H_*(R))$.

The $S$-algebra structure of $R$ then induces *Dyer–Lashof operations* $Q^i : H_n R \longrightarrow H_{n+i} R$ defined by

$$Q^i(x) = \xi_* (e_{i-n} \otimes x \otimes x).$$
Remarkable facts:

- all nonzero operations (any \( \pi \subset \Sigma_n \) for any \( n \)) can be written in terms of the \( Q^i \).
- all relations between them are generated by the natural map

\[
D_{\Sigma p} D_{\Sigma p} R \cong D_{\Sigma p} \Sigma_p R \longrightarrow D_{\Sigma p^2} R
\]

This is worked out nicely in J. Peter May’s ‘A General Algebraic Approach to Steenrod Operations’ (Springer LNM V. 168).

Let \( Ch \) be the category of \( \mathbb{F}_p \) chain complexes and \( Sym_1 \) the category of \( \mathbb{F}_p \) chain complexes with a homotopy associative product.
May defines a category $\text{Sym}_\infty$ and functors

$$\begin{align*}
\text{Top} & \xrightarrow{C^*} C_{\text{cocommHopf}} & \longleftarrow & \text{Sym}_\infty & \xrightarrow{H} & \mathcal{A} - \text{algebras} \\
\infty - \text{LoopSp} & \xrightarrow{C_*} \text{LoopSp} & \xrightarrow{C_*} & \text{Sym}_1 & \xrightarrow{H} & \text{Graded } \mathbf{F}_p - \text{algebras} \\
\text{LoopSp} & \xrightarrow{C_*} \text{Sym}_1 & \xrightarrow{C_*} & \text{Ch} & \xrightarrow{H} & \text{Graded } \mathbf{F}_p - \text{modules}
\end{align*}$$

where $\mathcal{A}$ is a universal Steenrod algebra.
S-algebras can be treated more directly, as above, but fitting them into this picture through their (cellular) chains elucidates the origin of the properties of the Dyer–Lashof operations.

\( A \) has as quotients

- the Dyer–Lashof algebra acting on \( \infty \)-loop spaces,
- the usual Steenrod algebra acting on the cohomology of topological spaces (sSets)
- an extension of the usual Steenrod algebra in which \( Sq^0 \neq 1 \) acting on the cohomology of cocommutative Hopf algebras.

The last of these acts more generally on \( \text{Ext}_C(M, N) \) when

- \( C \) is a cocommutative Hopf algebra,
- \( M \) is a cocommutative \( C \)-coalgebra and
- \( N \) is a commutative \( C \)-algebra.
Sym$_\infty$ consists of the Cartan and Adem objects in a category $\mathcal{C}(C_p, F_p)$, with

objects $(K, \theta)$:

- $K$ a $\mathbb{Z}$–graded homotopy associative differential $F_p$-algebra
- $\theta : \mathcal{W} \otimes K^p \rightarrow K$ a morphism of $F_p[C_p]$-complexes, satisfying
  - $\theta|\langle e_0 \rangle \otimes K^p$ is the $p$-fold iterated product associated in some fixed order, and
  - $\theta$ is $F_p[C_p]$-homotopic to a composite

\[
\mathcal{W} \otimes K^p \rightarrow \mathcal{V} \otimes K^p \xrightarrow{\phi} K
\]

for some $F_p[\Sigma_p]$-resolution $\mathcal{V}$ of $F_p$ and some $F_p[\Sigma_p]$-morphism $\phi$. 

Robert Bruner (Wayne State University) 
Ring Spectra - Hamburg
morphisms \( K \xrightarrow{f} L \), a morphism of \( F_p \)-complexes such that

\[
\begin{array}{c}
\mathcal{W} \otimes K^p \xrightarrow{\theta} K \\
1 \otimes f^p \downarrow \downarrow \downarrow f \\
\mathcal{W} \otimes L^p \xrightarrow{\theta'} L
\end{array}
\]

is \( F_p[C_p] \)-homotopy commutative.

Then

\[
Q^i(x) = \theta_*(e_{i-n} \otimes x \otimes x)
\]

defines \( Q^i : H_n(K) \longrightarrow H_{n+i}(K) \) if \( p = 2 \), and similarly for odd \( p \). Of course, they will not have many desirable properties without additional structure.
Cartan objects

With an evident tensor product in $\mathcal{C}(C_p, F_p)$ induced by the diagonal $\mathcal{W} \rightarrow \mathcal{W} \otimes \mathcal{W}$, we say that $(K, \theta)$ is a Cartan object if the product $(K, \theta) \otimes (K, \otimes) \rightarrow (K, \theta)$ is a morphism.

If $(K, \theta)$ is a Cartan object, the operations in $H(K)$ satisfy the Cartan formula.
Adem objects

Let $\mathcal{Y}$ be an $\mathbf{F}_p[\Sigma p^2]$-resolution of $\mathbf{F}_p$. Say that $(K, \theta)$ is an Adem object if there exists a $\Sigma p^2$-equivariant $\phi : \mathcal{Y} \otimes K p^2 \rightarrow K$ such that

$$
(W \otimes W^p) \otimes K p^2 \xrightarrow{w \otimes 1} \mathcal{Y} \otimes K p^2
$$

is $C_p \wr C_p$-equivariantly homotopy commutative.

If $(K, \theta)$ is an Adem object, the operations in $H(K)$ satisfy the Adem relations.
(This section is joint work with John Rognes.)

Let $\mathbb{T}$ be the circle group $S(\mathcal{C})$.

Let $R$ be a $\mathbb{T}$-equivariant commutative $S$-algebra.

E.G., $THH(B)$ for a commutative $S$-algebra $B$.

Then

$$R^{h\mathbb{T}} = F(E\mathbb{T}_+, R)^\mathbb{T}$$

is again an $S$-algebra, as are the terms in the limit system

$$
\cdots \longrightarrow F(S(\mathcal{C}^n)_+, R)^\mathbb{T} \longrightarrow F(S(\mathcal{C}^{n-1})_+, R)^\mathbb{T} \longrightarrow \cdots \longrightarrow R
$$
The Homotopy Fixed Point Spectral Sequence

**Theorem**

There is a natural $A_\ast$-comodule algebra spectral sequence

$$E^{2*}_*(R) = H_{gp}^{-\ast}(\mathbb{T}; H_\ast(R; F_p)) = P(y) \otimes H_\ast(R; F_p)$$

with $y$ in bidegree $(-2, 0)$, converging conditionally to the continuous homology

$$H^c_\ast(R^h\mathbb{T}; F_p) = \lim_n H_\ast(F(S(C^n)_+, R)^\mathbb{T}; F_p)$$

of the homotopy fixed point spectrum $R^h\mathbb{T} = F(E\mathbb{T}_+, R)^\mathbb{T}$. If $H_\ast(R; F_p)$ is finite in each degree, or the spectral sequence collapses at a finite stage, then the spectral sequence is strongly convergent.

**Theorem**

There are natural Dyer–Lashof operations $\beta^\epsilon Q^i$ acting vertically on this spectral sequence, and they commute with the differentials:

$$d^{2r}(\beta^\epsilon Q^i(x)) = \beta^\epsilon Q^i(d^{2r}(x))$$
Propagating differentials

Given a $d^{2r}$, write $d^{2r} x = y^r \delta x$:

$$t + 2r - 1 : y^r \delta x$$

This is

$$x : S(C)^+ \wedge S^t \longrightarrow H \wedge R$$

which extends to

$$x' : S(C^r)^+ \wedge S^t \longrightarrow H \wedge R.$$ 

and induces

$$H \wedge D_p(S(C^r)^+ \wedge S^t) \longrightarrow H \wedge D_p(H \wedge R) \longrightarrow H \wedge H \wedge R \longrightarrow H \wedge R.$$
In the domain, and hence in the codomain, we get $d^{2r}$s:

$$y^r Q^{t+2r} \delta x$$

$$y^r Q^{t+2r-1} \delta x$$

$$\vdots$$

$$Q^{t+2r} x$$

$$Q^{t+2r-1} x$$

$$\vdots$$

$$Q^{t+2} x$$

$$Q^{t+1} x$$

$$Q^t x$$
but also,

\[
\begin{align*}
&y^r Q^{t+2r} \delta x \\
&y^r Q^{t+2r-1} \delta x \\
&y^r Q^{t+2r-2} \delta x \\
&\vdots
\end{align*}
\]
The wonderful thing here is that these classes don’t just survive to $E^{2r+1}$.

**Theorem**

The elements $Q^t x, Q^{t+1} x, \ldots, Q^{t+2r-2} x$ and $Q^{t+2r-1} x - x \delta x$ are all infinite cycles.
Proof: At $E^{2r+1}$ in the domain spectral sequence we have nonzero classes only in columns $0$ to $2r - 2$ and rows $2t$ to $2t + 2r - 2$: 
Applications

- RRB and Rognes [A&GT, V. 5] show that the spectral sequence for $H^c_* THH(B)^{hT}$ collapses at $E^4$ for $B = MU, BP, ku, ko, \text{ and } tmf$, and calculate the result.

- Caruso, May and Priddy, [Topology V. 26], show that the continuous homology serves as input to an Adams spectral sequence for the homotopy of an inverse limit.

- Similar results hold for $R^{hC}$ for cyclic $C \subset \mathbb{T}$ and for the analogous spectral sequences for the homotopy orbits $R_{hC}$ and $R_{h\mathbb{T}}$ and Tate spectra $R^{tC}$ and $R^{t\mathbb{T}}$.

- Sverre Lunøe-Nielsen’s work computes the $A_*$-comodule structure, with $K(B)$ as the intended goal.
Historically, this came first (late 1960s to 1970s) in the work of D. S. Kahn, R. J. Milgram, J. Mäkinen and RRB.

In the abutment, we have homotopy operations compatible with the Dyer-Lashof operations in homology under the Hurewicz map.

Between $E_2$ and $E_\infty$ we have Steenrod operations in Ext interpolating between these.

Two extremes of power operations in other cohomology theories, e.g., in the work of N. P. Strickland, C. Rezk, and T. Torii.
Homotopy operations

\[ S^n \xrightarrow{x} R \]

\[ D_G S^n \xrightarrow{D_G x} D_G R \xrightarrow{\xi} R \]

\[ S^k \xrightarrow{\alpha} \]

\[ \alpha^*(x) \]
**Cup-i operations**

We call the operation ‘cup-i’

\[ S^n \xrightarrow{x} R \]

\[ D_2 S^n \xrightarrow{D_2 x} D_2 R \xrightarrow{\xi} R \]

if

\[ \cup_i \in \pi_{2n+i} D_2 S^n = \pi_{2n+i} \Sigma^n P_n \]

\[ \cup_i \in \text{gen} \]

\[ \Rightarrow \]

\[ H_{2n+i} D_2 S^n \]
Detection in the Adams spectral sequence

The cohomology of a cocommutative Hopf algebra, such as the Steenrod algebra, has natural operations

\[ Sq^i : \text{Ext}_A^{s,t}(H^* R, F_2) \longrightarrow \text{Ext}_A^{s+i,2t}(H^* R, F_2) \]

for \( 0 \leq i \leq s \) in the cohomological indexing, or

\[ Q^i : \text{Ext}_A^{s,t}(H^* R, F_2) \longrightarrow \text{Ext}_A^{s+t-i,2t}(H^* R, F_2) \]

for \( t - s \leq i \leq t \) in the homological indexing.
Cohomological indexing:

\[ \text{Sq}^s x \]

\[ \text{Sq}^{s-1} x \]

\[ \text{Sq}^0 x \]

\[ \text{Sq}^i : \text{Ext}^{s,t} \rightarrow \text{Ext}^{s+i,2t} \quad (n = t - s) \]
Homological indexing:

\[ Q^n x \]

\[ Q^{n+1} x \]

\[ Q^t x \]

\[ Q^i : \text{Ext}^{s,t} \longrightarrow \text{Ext}^{s+t-i,2t} \ (n = t - s) \]
Properties of the cup-i operations

- $\cup_0(x) = x^2$ and always exists.
- $\cup_i : \pi_n \to \pi_{2n+i}$ is detected by $Q^{n+i} = Sq^{s-i}$ in Ext.
- Each cell of $D_2S^n$ either defines a $\cup_i$ operation or a relation between lower operations.
- For example, $\cup_1 : \pi_n \to \pi_{2n+1}$ exists iff $n$ is even.
- If $n$ is odd then the $2n + 1$ cell of $D_2S^n = \Sigma^n P_n$ instead gives a null-homotopy of $2x^2$. 
Manifestation in the Adams spectral sequence

Let

\[ R \leftarrow R_1 \leftarrow R_2 \leftarrow \cdots \]

be an Adams resolution of \( R \). Taking \( p \)-fold smash product, the comparison theorem gives us a map

\[ D_p R \]

\[ R^{(p)} \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots \]
In $A_*$-comodules this gives maps of resolutions

$$
\begin{align*}
0 & \longrightarrow H_\ast R & \longrightarrow & C_\ast \\
\uparrow & & \uparrow & \quad C_i = H_\ast(\Sigma^i(R_i/R_{i+1})) \\
0 & \longrightarrow H_\ast R^{(p)} & \longrightarrow & C^{(p)}
\end{align*}
$$

Standard homological algebra then extends this map of resolutions to a $\Sigma_p$-equivariant homomorphism

$$
\mathcal{W}_i \otimes C_s^{(p)} \longrightarrow C_{s-i}.
$$

The spectrum $(E\Sigma_p)_+ \wedge R^{(p)}$ is bifiltered by the $(E\Sigma_p)^i_+ \wedge B_s$ with filtration quotients $\mathcal{W}_i \otimes (C^{(p)})_s$. 
When $R$ is an $S$-algebra, this algebraic map allows us to filter the structure map $D_p R \to R$ to give compatible maps

$$(E \Sigma_p)^i \wedge B_s \to R_{s-i}$$

genuinely realizing the Steenrod operations in Ext. This is how we detect the homotopy operations in the Adams spectral sequence. First suppose we have a permanent cycle

$$S^n \downarrow \quad x$$

$$R \leftarrow \cdots \leftarrow R_s$$
Applying the extended powers, the characteristic map of the $2n + i$-cell of $D_2 S^n$ ‘carries’ $Q^{n+i}(x)$:

\[
\begin{array}{cccc}
  e^{2n+i} & \leftarrow & S^{2n+i-1} \\
  \downarrow & & \downarrow \\
  \Sigma^n P_n^{n+i} & \leftarrow & \Sigma^n P_n^{n+i-1} \\
  \downarrow & & \downarrow \\
  R_{2s-i} & \leftarrow & R_{2s-i+1}
\end{array}
\]
For example, if $n$ is odd then $P_{n}^{n+1} = S^{n} \cup_{2} e^{n+1}$ and we have

\[
\begin{array}{c}
e^{2n+1} \leftarrow S^{2n} \\
\downarrow \\
\sum_{n} P_{n}^{n+1} \leftarrow S^{2n} \\
\downarrow \\
R_{2s-1} \leftarrow R_{2s} \\
\downarrow \\
R_{2s+1}
\end{array}
\]

resulting in the differential $d_{2}(\cup_{1}(x)) = h_{0}x^{2}$. 
Cup-1 of 2 is $\eta$

Consider operations on $2 \in \pi_0 S$.

\[
\begin{array}{ccccccc}
D_2S & \xrightarrow{D_2^2} & D_2S & \xrightarrow{\xi} & S \\
\uparrow \cup_1 & & & & & \\
S^1 & \xrightarrow{\eta} & & & & \\
\end{array}
\]

This is detected by $Sq^0(h_0) = h_1$ in $\text{Ext}_A(F_2, F_2)$. 
Cup-1 of $\eta$ is not defined

However, we do have $Sq^0(h_1) = h_2$ in Ext.
Restricting to the 3-skeleton,
The attaching map of the 3-cell of $\Sigma P_1$ has degree 2, giving the Adams spectral sequence differential $d_2(h_2) = h_0 h_1^2 = 0$. There are no possible higher differentials, allowing $\nu$ to exist.
Similarly,

$$D^1_2 S^3 \xrightarrow{D_2 \nu} D^1_2 S \xrightarrow{\xi} S$$

Again, the attaching map has degree 2, and this gives $d_2(h_3) = h_0 h_2^2 = 0$, and there are no possible higher differentials, allowing $\sigma$ to exist as well.
• After this, the differential $d_2(h_{n+1}) = h_0 h_n^2 \neq 0$, and no higher Hopf maps exist.

• In this sense, $\eta$ must exist, while $\nu$ and $\sigma$ are 'gifts', or low dimensional accidents.

• The 15 cell carrying $h_4$ is a null-homotopy of $2\sigma^2$, showing that $2\theta_3 = 0$.

• For higher $n$, we don't get the implication $2\theta_n = 0$ from the differential $d_2(h_{n+1}) = h_0 h_n^2$, though, because $h_n$ was not a homotopy class to start with and the story is a bit more complicated.

• The boundary of the cell carrying $h_n$ decomposes into a part carrying $h_0 h_n^2$ and a part carrying operations on $h_0 h_{n-1}^2$, effectively setting $2\theta_n$ equal to higher Adams filtration elements which we must analyze.
One more example. Suppose \( n \equiv 2 \pmod{4} \). Then
\[
\mathcal{P}_n^{n+2} = (S^n \lor S^{n+1}) \cup_{(\eta,2)} e^{n+2}.
\]
In the Adams spectral sequence this manifests as

\[
\begin{array}{c}
h_1 x^2 \\ h_0 \cup_1 (x) \\ \cup_1 (x)
\end{array}
\]
\[
\begin{array}{c}
h_1 x^2 \\ h_0 \cup_1 (x) \\ \cup_1 (x)
\end{array}
\]

The \( d_2(\cup_2(x)) = h_0 \cup_1 (x) \) here reflects the relation

\[
2 \cup_1 (x) + \eta x^2 = 0
\]

In the Adams spectral sequence this is a ‘universally hidden extension’:
At $E_\infty$, if $x \in E_\infty$ is in a stem $\equiv 2 \pmod{4}$, we have

$$h_1x^2 \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downar
More generally, if we start with $x \in E_r$, it is realized geometrically by

\[
\begin{array}{c}
  e^n \\ \downarrow x \\
  R_s \\
  \end{array} \rightarrow \begin{array}{c}
  S^{n-1} \\ \downarrow d_r x \\
  R_{s+r} \\
  \end{array}
\]

The smash square of a pair $e^n \supset S^{n-1}$ is a $\Sigma_2$-equivariant filtration of length 3

$$e^n \wedge e^n \supset e^n \wedge S^{n-1} \cup S^{n-1} \wedge e^n \supset S^{n-1} \wedge S^{n-1}$$

which we abbreviate to $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2$. 
The boundary of the top cell of \((E\Sigma_2)^i\) decomposes into two pieces:
- one carries a lower operation on \(x\) itself (using \(\Gamma_0\)), while
- the other carries an operation on \(d_r x\) (using \(\Gamma_2\)).

This results in

\[
d_\ast Sq^i x = Sq^{i+r-1} d_r x + \begin{cases} 
a_j Sq^{i+j} x & j \leq s - i 
a_j x d_r x & j = s - i + 1 
0 & j > s - i + 1 \end{cases}
\]

where \(j\) is the vector fields number for \(P^{n+s-i}\), telling how far its top cell compresses.
• May’s theorem on Steenrod operations in spectral sequences derived from filtrations.

• Phil Hackney’s thesis on operations in the homology of a cosimplicial $E_\infty$-space and Jim Turner’s earlier work.

• Kristine Bauer and Laura Scull’s results on preservation of operad actions in spectral sequences.

• General idea: S-algebra structures produce operations, differentials, and hidden extensions in spectral sequences. Sean Tilson is working this out for the Kunneth spectral sequence, as we speak, as part of his thesis.
Thank you