TWO GENERALIZATIONS OF THE ADAMS SPECTRAL SEQUENCE

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By an ordinary Adams spectral sequence with respect to a spectrum $E$, we shall mean the kind of spectral sequence described by Adams in [Al, Part III, Chap. 14]. Note that Adams does not assume $E_*X$ is projective in order to construct the spectral sequence, but only to get an algebraic description of $E_2$ in terms of $E$ homology. In detail, let $Y$ be a spectrum and let

$$Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

be an inverse sequence such that if we let $\overline{Y_i} = Y_i/Y_{i+1}$ then

(i) $Y = Y_0$

(ii) $E_*\overline{Y_i} \rightarrow E_*\overline{Y_1}$ is a $\pi_*E$ split monomorphism

and (iii) $\overline{Y_i}$ is a retract of a spectrum $\overline{Y_1} \wedge E$ for some $\overline{Y_i}$.

By applying $[X,\_]_* = \pi_*F(X,\_)$ to (1) we obtain an exact couple and, hence, a spectral sequence. As usual, $E_1$ depends on the particular resolution (1) chosen, while $E_r$ for $r > 1$ depends only on $X$ and $Y$. To indicate this dependence we adopt the following

NOTATION. Let $E_{st}^1(X,Y_i) = [X,\overline{Y_i}]_{t-s}$. For $r \geq 2$ let $E_{st}^r(X,Y_i) = E_{st}^r(X,[Y_i])$ be the usual subquotient of $E_{st}^1(X,[Y_i])$.

Under appropriate hypotheses this spectral sequence assumes the particularly pleasing form

$$E_{st}^2 = \text{Ext}^E_{E_*E}(E_*X,E_*Y) = [X,Y]^E_{t-s},$$

where the Ext is that of $E_*E$ comodules relative to $\pi_*E$ split exact sequences, and $[\_\_\_]^E_\pi$ denotes homotopy classes of morphisms in the stable category localized at $E$.

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The two generalizations of the Adams spectral sequence referred to in the
title are obtained by mixing (1) with either a direct or inverse sequence.
(The reader is warned that we do not prove convergence here. For our applica-
tions, Propositions 8 and 9, convergence is irrelevant. See the penultimate
paragraph for further remarks on convergence.) To proceed, suppose given
direct and inverse sequences

(2) \[ X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots \], and
(3) \[ Z = Z_0 \leftarrow Z_1 \leftarrow Z_2 \leftarrow \ldots \].

Then \( F = F(X_0, Y_0) \) and \( W = Z_0 \wedge Y_0 \) are bifiltered by \( F_{ij} = F(X_i, Y_j) \) and
\( W_{ij} = Z_i \wedge Y_j \). By [R2, Lemma 3.1], we may suppose each inclusion in these bi-
filtrations is the inclusion of a subcomplex. Then, if we totalize them by
setting

\[ F_n = \bigcup_{i+j=n} F_{ij} \] and \[ W_n = \bigcup_{i+j=n} W_{ij} \]

and we let

\[ \overline{X}_i = \sum^{-1} (X_{i+1}/X_i), \quad \overline{Z}_i = Z_i/Z_{i+1}, \]
\[ \overline{F}_n = F_n/F_{n+1}, \] and \[ \overline{W}_n = W_n/W_{n+1}, \]

we have the relations

\[ \overline{F}_n = \bigvee_{i+j=n} F(X_i, Y_j) \] and \[ \overline{W}_n = \bigvee_{i+j=n} Z_i \wedge Y_j. \]

Furthermore, the boundary maps \( \delta : \overline{F}_n \rightarrow \overline{F}_{n+1} \) and \( \delta : \overline{W}_n \rightarrow \overline{W}_{n+1} \) are
converted by these equivalences to

\[ \bigvee_{i+j=n} F(\delta, 1) + F(1, \delta) \] and \[ \bigvee_{i+j=n} \delta + 1 + 1 \wedge \delta. \]

By applying \([A, -]_\delta \) to the sequences

\[ F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \]

and

\[ W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \ldots \]

we obtain spectral sequences which we shall denote by \( \tilde{E}_r \) and \( \tilde{E}_r \)
respectively.

The following results are standard.
PROPOSITION 1: (a) \( E^s_1 = \oplus_{i=1}^s f_i^{s-1,t-1}(A \wedge \overline{X}_i, \{y_j\}) \)
\( E^s_1 = \oplus_{i=1}^s f_i^{s-1,t-1}(A, \overline{Z}_i \wedge \{y_j\}) \)

(b) The spectral sequences are natural with respect to
\( A, \{X_i\}, \{Y_j\} \) and \( \{Z_k\} \).

(c) If they converge, then
\( E^*_{\infty} = [A, F(X,Y)]_{t-s} = [A \wedge X, Y]_{t-s} \)
\( E^*_{\infty} \to [A, Z \wedge Y]_{t-s} \).

PROPOSITION 2: There are smash product pairings
\( E^*_{\infty}(A, [F(X_i, Y_j)]) \oplus E^*_{\infty}(A', [F(X'_i, Y'_j)]) \to E^*_{\infty}(A \wedge A', [F(X_i \wedge X'_i, Y_j \wedge Y'_j)]) \)
\( \downarrow \)
\( [A, F(X,Y)]_{t-s} \oplus [A', F(X', Y')]_{t-s} \to [A \wedge A', F(X \wedge X', Y \wedge Y')]_{t-s} \)

and
\( E^*_{\infty}(A, [Z_i \wedge Y_j]) \oplus E^*_{\infty}(A', [Z'_i \wedge Y'_j]) \to E^*_{\infty}(A \wedge A', [Z_i \wedge Z'_i \wedge Y_j \wedge Y'_j]) \)
\( \downarrow \)
\( [A, Z \wedge Y]_{t-s} \oplus [A', Z' \wedge Y']_{t-s} \to [A \wedge A', Z \wedge Z' \wedge Y \wedge Y']_{t-s} \)

Proposition 2 implies that \( \overline{E}^*_{\infty} \) and \( \overline{E}^*_{\infty} \) are modules over the spectral sequence \( \text{Ext}_{E^*}_{x,y}(\pi_{\infty}E, \pi_{\infty}E) = \pi_{\infty}S \).

Note that if we use trivial direct and inverse systems
\( X \to * \to * \to \cdots \) or \( Z \leftarrow * \leftarrow * \leftarrow \cdots \)
then \( \overline{E}^*_{\infty} \) and \( \overline{E}^*_{\infty} \) become ordinary Adams spectral sequences.

A spectral sequence is most useful when one can identify its starting point. Toward this end we introduce the following algebraic lemma, which we learned from Doug Ravenel.

Let the sequence of \( R \)-modules (\( R \) commutative with 1)
\( \begin{array}{c}
0 \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \\
0 \to I_0 \to I_1 \to \cdots
\end{array} \)
and
\( 0 \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots \)
be exact, and let \( P_\alpha, I_\alpha \) and \( Q_\alpha \) denote the complexes with the first term (F, I, or Q) omitted. Call a complex projective, injective, or flat if each term is. If \( C = \{ C_{ij} \} \) is a bicomplex, let \( \text{Tot}(C) \) be the complex with

\[
\text{Tot}(C) = \oplus_{i+j=n} C_{ij}.
\]

**Lemma 3.** If \( P_\alpha \) is projective or \( I_\alpha \) is injective then

\[
H(\text{Tot}(\text{Hom}(P_\alpha, I_\alpha))) = \text{Ext}(P, I).
\]

If either \( P_\alpha \) or \( Q_\alpha \) is flat then

\[
H(\text{Tot}(P_\alpha \otimes Q_\alpha)) = \text{Tor}(P, Q).
\]

**Proof.** In each case the result follows from the collapse of an appropriate bicomplex spectral sequence.//

To apply this lemma to the calculation of \( \widetilde{E}_2 \) and \( \widetilde{E}_2 \) we associate complexes \( P_\alpha, I_\alpha \), and \( J_\alpha \) to the sequences (1) - (3) by setting

\[
\begin{align*}
I &= E_\alpha Y \\
I_1 &= E_\alpha \Sigma^1 Y_1 \\
J &= E_\alpha Z \\
J_1 &= E_\alpha \Sigma^1 Z_1 \\
P &= E_\alpha X \\
P_1 &= E_\alpha \Sigma^{-1} X_1,
\end{align*}
\]

with differentials defined in the usual way from the sequences (1) - (3).

Let us call a complex trivial if each differential in it is 0.

**Lemma 4.** (a) If each map in a sequence (1), (2) or (3) induces a monomorphism in \( E \) homology, the associated complex is trivial. 

(b) If each map in the sequence induces an epimorphism in \( E \) homology, the associated complex is trivial.

(c) If each map in the sequence induces the zero homomorphism in \( E \) homology, the associated complex is exact.

In order to apply these algebraic lemmas we must be able to replace \([X, Y] \) by \( \text{Hom}_{E_\alpha E}(E_\alpha X, E_\alpha Y) \) for appropriate \( X \) and \( Y \). To this end we assume
(i) $E^*_\pi E$ is $\pi^*_\pi E$ projective

(ii) if $E^*_\pi A$ is $\pi^*_\pi E$ projective then, for any $Y$,

$$[A,Y^\wedge E] \cong \text{Hom}_{E^*_\pi E}(E^*_\pi A, E^*_\pi (Y^\wedge E)),$$

the isomorphism being given by taking the induced map in homology. Adams [Al, pp. 280 and 284ff] gives general conditions under which (ii) is satisfied, and shows they hold when $E$ is $S$, $HZ_p$, $MO$, $MU$, $MS_p$, $K$ or $KO$. Note that, by standard arguments, (ii) can be reduced to

(ii') If $E^*_\pi A$ is $\pi^*_\pi E$ projective then

$$E^*_\pi \vee (E^*_\pi A)^* = \text{Hom}_{\pi^*_\pi E}(E^*_\pi A, \pi^*_\pi E).$$

In what follows, let $E = \mathbb{E}$, let $\text{Hom} = \text{Hom}_{E^*_\pi E}$ be the graded module of $E^*_\pi E$ comodule homomorphisms, and let $\text{Ext}^1$ be the 1st derived functor of $\text{Hom}$ relative to the injective class generated by extended comodules. (These are injective relative to $\pi^*_\pi E$ split exact sequences.) Totalization will either add the degrees of the two complexes involved (as in $\text{Tot}(I^*_\pi \otimes J^*_\pi)$) or will add the homological degree to the degree of the sole complex involved (as in $\text{Tot}(\text{Ext}(E^*_\pi A \otimes F^*_\pi, E^*_\pi Y))$, whichever is appropriate. Internal degrees are not affected.

**Theorem 5.** (1) If $E^*_\pi A$ and $F^*_\pi$ are $\pi^*_\pi E$ projective then

$$\widetilde{E}_1 = \text{Tot}(\text{Hom}(E^*_\pi A \otimes F^*_\pi, I^*_\pi))$$

and (a) if, in addition, $F^*_\pi$ is trivial then

$$\widetilde{E}_2 = \text{Tot}(\text{Ext}(E^*_\pi A \otimes F^*_\pi, E^*_\pi Y)),$$

while (b) if, in addition, $F^*_\pi$ is exact then

$$\widetilde{E}_2' = \text{Ext}(E^*_\pi A \otimes E^*_\pi X, E^*_\pi Y).$$

(2) If $E^*_\pi A$ and either $J^*_\pi$ or $I^*_\pi$ are $\pi^*_\pi E$ projective then

$$\widetilde{E}_1 = \text{Tot}(\text{Hom}(E^*_\pi A, I^*_\pi \otimes J^*_\pi))$$

$$= \text{Hom}(E^*_\pi A, \text{Tot}(I^*_\pi \otimes J^*_\pi))$$
and (a) if, in addition, $J_*$ is trivial then

$$\tilde{E}_2 = \text{Tot}(\text{Ext}(E_*A,J_* \otimes E_*Y))$$

while (b) if, in addition, $J_*$ is $\pi_*E$ split exact then

$$\tilde{E}_2 = \text{Ext}(E_*A,E_*Z \otimes E_*Y).$$

**PROOF.** Since $E_*A$ and $E_*Z_1$ are $\pi_*E$ projective in part (1), there is a Kunneth isomorphism $E_*(A \wedge \overline{X}_1) \cong E_2^A \otimes E_*Z_1$, which implies that $E_*(A \wedge \overline{X}_1)$ is also $\pi_*E$ projective. Therefore,

$$[A,F(\overline{X}_1,\overline{Y}_j)] \cong [A \wedge \overline{X}_1,\overline{Y}_j]$$

$$\cong \text{Hom}(E_*A \otimes E_*Z_1, E_*\overline{Y}_j).$$

It follows that

$$\tilde{E}_1^{st} = \bigoplus_{i+j=s} [A,F(\overline{X}_1,\overline{Y}_j)]_{i+j}$$

$$\cong \text{Hom}^\ell(E_*A \otimes E_*\overline{X}_1, E_*\overline{Y}_j)$$

$$\cong \text{Hom}^\ell(E_*A \otimes P_1, I_j),$$

showing that $\tilde{E}_1 = \text{Tot}(\text{Hom}(E_*A \otimes P_*, I_*))$. If $P_*$ is trivial then $d_1 = \text{Hom}(1,d)$ and hence $\tilde{E}_2 = \text{Tot}(\text{Ext}(E_*A \otimes P_*, E_*Y))$. If $P_*$ is exact then Lemma 3 implies that $\tilde{E}_2 = \text{Ext}(E_*A \otimes E_*X, E_*Y)$.

In part (2), first note that since $\overline{Y}_j$ is a retract of some $\overline{Y}_j \wedge E$, $\overline{Z}_1 \wedge \overline{Y}_j$ is a retract of $\overline{Z}_1 \wedge \overline{Y}_j \wedge E$. Thus $[A,\overline{Z}_1 \wedge \overline{Y}_j] = \text{Hom}(E_*A,E_*\overline{X}_1 \wedge E_*\overline{Y}_j)$. Therefore

$$\tilde{E}_1^{st} = \bigoplus_{i+j=s} [A,\overline{Z}_1 \wedge \overline{Y}_j]_{i+j}$$

$$\cong \bigoplus_{i+j=s} \text{Hom}^\ell(E_*A,E_*\overline{Z}_1 \wedge E_*\overline{Y}_j)$$

$$\cong \bigoplus_{i+j=s} \text{Hom}^\ell(E_*A,J_1 \otimes I_j),$$

showing that $\tilde{E}_1 = \text{Tot}(\text{Hom}(E_*A,J_* \otimes I_*)) = \text{Hom}(E_*A,\text{Tot}(J_* \otimes I_*))$. If $J_*$ is trivial $\tilde{E}_2 = \bigoplus_1 \text{Hom}(E_*A,J_1 \otimes I_*))$ and, since $J_1 \otimes I_*$ is a $\pi_*E$ split.
injective resolution of $J_* \otimes E_* Y$,

$$\tilde{E}_2' = \oplus \text{Ext}(E_* A, J_* \otimes E_* Y) = \text{Tot}(\text{Ext}(E_* A, J_* \otimes E_* Y)).$$

If $J_*$ is $\pi_* E$ split, the bicomplex spectral sequence

$$H^{\text{II}} H^1 \text{Hom}(E_* A, J_* \otimes I_*) \Rightarrow H(\text{Tot}(\text{Hom}(E_* A, J_* \otimes I_*))) = \tilde{E}_2'$$

collapses to an isomorphism $\tilde{E}_2' \cong \text{Ext}(E_* A, E_* Y \otimes E_* Y)$ if we let

$d_1 = \text{Hom}(1, d \otimes 1)$ and $d_{\text{II}} = \text{Hom}(1, 1 \otimes d).$ Note that because $I_*$ is injective, we need not assume $J_*$ is. //

COROLLARY 6. There is a spectral sequence

$$\text{Ext}^s_{E_* E_*} (F_* X, F_* Y) \Rightarrow [X, Y]^{E_*}_{t=s}.$$ 

PROOF. The difference between this and the ordinary Adams spectral sequence is that we have not assumed $E_* X$ is $\pi_* E$ projective. Instead, construct a direct sequence $\{X_i\}$ for which $P_*$ is a $\pi_* E$ projective resolution of $E_* X$, as in [A1, Thm.13.6]. Then Theorem 5.1(b) shows that, with $A=S$, the spectral sequence $\tilde{E}_r$ has the desired form. In fact we get a bit more. The map of direct systems

$$X \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

induces a homomorphism from the ordinary Adams spectral sequence into $\tilde{E}_r$ converging to $1 : [X, Y]^{E_*} \rightarrow [X, Y]^{E_*}$. If $E_* X$ is $\pi_* E$ projective this is an isomorphism from $E_* Y$ on. In general, it shows that we get the "right" $E_2$ term at the cost of allowing elements to have higher filtration. //

This Corollary is by no means new. In particular, it was known to Frank Adams by 1968 or 1969 [private communication]. He summarizes the method of construction by the appealing slogan (paraphrased): resolve $E_* X$ by $\pi_* E$ projectives, resolve $E_* Y$ by (relative) $E_* E$ injectives, and mix the resolutions geometrically. These ideas can be found in [A2, pp.50 and 54].
When $E_nX$ is $\pi_nE$ projective, the isomorphism between the two spectral sequences (from $E_2$ on) has the following useful consequence. If we wish to construct a homomorphism

$$\tilde{E}_r([X'_1], [Y'_j]) \rightarrow E_r(X,Y)$$

converging to the homomorphism $\tau^* : [X', Y] \rightarrow [X, Y]$ induced by a map $f : X \rightarrow X'$, it suffices to construct a map of direct systems

$$\begin{array}{cccc}
X & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & \cdots \\
\downarrow f & & \downarrow & & \downarrow & & \\
X' & \rightarrow & X'_1 & \rightarrow & X'_2 & \rightarrow & \cdots
\end{array}$$

for any direct system $\{X'_1\}$ whose associated complex $P^*$ is exact. Roughly, instead of factoring $X \rightarrow X' \rightarrow X'_1$ through the $0$ map, it suffices to factor it through the composite of $i$ maps, each of which induces the $0$ homomorphism in $E$ homology. Ravenel uses this feature of $\tilde{E}_r$ very effectively in his proof of the Segal conjecture for $\mathbb{Z}_{p^n}$ [R1]. This is the dual of a result which Milgram [M2, M3] and the author [B1, B2] have used to identify differentials in the Adams spectral sequence for $\pi_*S$ or, more generally, $\pi_*Y$ for any $H_\infty$ ring spectrum $Y$. We state that result in the following proposition. Let

$$S \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots$$

be an Adams resolution of $S$.

**COROLLARY 7.** A map $\{f'_i\} : \{Z'_i\} \rightarrow \{Y'_i\}$ from an inverse sequence to an Adams resolution induces a homomorphism

$$\tilde{E}_r(A, [Z'_i \wedge S_j]) \rightarrow E_r(A, Y)$$

$$[A, E](f'_i)_* \rightarrow [A, Y]$$

**PROOF.** This follows, by naturality of $\tilde{E}_r$, from the isomorphism

(for $r \geq 2$) $E_r(A, \{Y'_i\}) \rightarrow \tilde{E}_r(A, \{Y'_i \wedge S_j\})$ induced by the maps

$$Y' \leftarrow Y \wedge S \leftarrow \bigcup_{i+j=n} Y'_i \wedge S_j.$$  The isomorphism comes from the fact that the complexes associated to $\{Y'_i\}$ and $\{Y'_i \wedge S_j\}$ are both injective resolutions.
of $E_{\infty}$ and the map between them induces a chain homomorphism covering the identity of $Y$. //

We wish to apply this to find families of differentials in the $BP$ Adams spectral sequence for the homotopy of an $H_\infty$ ring spectrum (for example, the sphere spectrum). By a construction to appear in [B2] and similar to that in [M1, §11], there are Steenrod operations

$$
\beta^j: \text{Ext}^s \longrightarrow \text{Ext}^{sp - [(2j + 2)(p-1) - 1]} \text{tp}
$$

for $t - s < 2j \leq t$

if $\text{Ext} = \text{Ext}_{BP_*BP}(C, A)$ where $C$ is a coalgebra and $A$ is an algebra in the category of $BP_*BP$ comodules. If $C = BP_*S$ and $A = BP_*Y$, where $Y$ is an $H_\infty$ ring spectrum, we can construct a map representing $\beta^j$, given a map representing $x$. If $L^n$ is the $n$-skeleton of the usual $p$-localization of $\Sigma^\infty(BP_p)$ [A3], and $L_n$ is the quotient $L^n/L^{n-1}$, then for any $x \in \text{Ext}^s[n + s](BP_*BP_*Y)$ we obtain a map of inverse sequences

$$
\cdots \leftarrow \sum_n L_n^{n(p-1)+1} \leftarrow \cdots \leftarrow \sum_n L_n^{n(p-1)}
$$

(4)

$$
Y \leftarrow \cdots \leftarrow Y_{ps-1} \leftarrow \cdots \leftarrow Y_{ps}
$$

An appropriate multiple of the characteristic map of the $n + 2j(p-1) - 1$ cell represents $\beta^j$ [B2]. Thus, if $[Z_j]$ is the skeletal filtration of $\Sigma^\infty L_n^{n(p-1)}$ as in (4), Corollary 7 implies that differentials in $E_2(S, [Z_j])$ translate into differentials in $E_\infty(S, Y)$ relating the $\beta^j$.

This process of translation is begun in the following two results.

PROPOSITION 8. $E_2(S, [Z_j])$ contains elements $\beta^j$ in filtration $sp - [(2j+n)(p-1) - 1]$ and stem degree $n + 2j(p-1) - 1$ for each $j$ such that $n < 2j \leq n+1$. Each $\beta^j$ generates a copy of $E_2(S, S \cup e^1)$. In addition, if $n = 2k$ there is a generator $\phi$ in filtration $sp$ and stem degree $np$ which generates a copy of $E_2(S, S)$.
PROOF. Since L has a cell in each nonnegative dimension congruent to 0 or -1 \mod 2(p-1), it is easy to check that the complex $J^*$ associated to $\{Z^i\}$ contains $BP_S$ in each degree $s - (2j-n)(p-1) + \varepsilon$, $\varepsilon = 0$ or 1, in the range $sp$ to $s$ (or $s-1$). (We truncate $\{Z^i\}$ below filtration $s$ or $s-1$ since the Steenrod operations are trivial below filtration $s$. We truncate at $s-1$ rather than $s$ if this is necessary to complete a pair of copies of $BP_S$ in adjacent degrees.) It is well-known that the differential in $J^*$ is simply multiplication by $p$. Thus, $J^*$ consists of segments

$$0 \rightarrow BP_s \xrightarrow{d} BP_{s+1} \rightarrow 0$$

surrounded by 0's, together with a single copy of $BP_s$ in degree $ps$ when $n$ is even. Thus we need only show that if $J^ib$ is the sequence in (5) then

$$H(Tot(Hom(BP_S, J^ib \otimes I^b))) = Ext(BP_S, BP_S(S \cup e^i))$$

generated by the second $BP_S$ in (5). This follows easily from the bicomplex spectral sequence whose first differential is $d \otimes 1$. //

Let * denote equality up to multiplication by a nonzero constant.

PROPOSITION 9. $d_{2p-1} \alpha_{j+1} = \alpha_{j+1}$ if $j \neq -1 (p)$. 

PROOF. The $2(j+1)(p-1)-1$ cell is attached by $-(j+1)\alpha_1$ to the $2j(p-1)-1$ cell, where $\alpha_1$ is the Hopf map in $\pi_{2p-3}S$. This implies the differential $d_{2p-1} \alpha_{j+1} = \alpha_{j+1}$ in $BP_r$. By Corollary 7, it applies to $E_r(S,Y)$. //

Our final result shows that the duality between $\mathbb{K}_r$ and $\mathbb{K}$ becomes an isomorphism with appropriate finiteness restrictions. Let

$$Z \cong Z_0 \leftarrow Z_1 \leftarrow \cdots$$

be a sequence of finite complexes, and let

$$S \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots$$

be an Adams resolution of S. Then we obtain
\[ \widetilde{E}_r(s, \{z_i \wedge s_j\}) = \pi_* z. \]

Applying the functor \( D(X) = F(X, S) \) to \( \{z_i\} \) gives
\[ DZ_0 \rightarrow DZ_1 \rightarrow \cdots \]
from which we may construct
\[ \widetilde{E}_r(s, \{f(DZ_i, s_j)\}) = \pi_* DDZ. \]

In general, the maps
\[ z_i \wedge s_j \rightarrow F(DZ_i, s_j) \]
adjoint to evaluation \( DZ_i \wedge z_i \wedge s_j \rightarrow s_j \) induce a homomorphism
\[ \widetilde{E}_r \rightarrow \widetilde{E}_r \]
which converges to the homomorphism induced by the canonical map
\[ Z \rightarrow DDZ. \] Since all the \( z_i \) are finite complexes, the maps
\[ z_i \wedge s_j \rightarrow F(DZ_i, s_j) \]
are all equivalences, which means that the two spectral sequences are isomorphic. In fact, the same argument shows the following

**Proposition 10.** If \( \{z_i\} \) is an inverse sequence of finite complexes, there are isomorphisms
\[
\begin{align*}
\widetilde{E}_r\left(A, \{z_i \wedge y_j\}\right) & \xrightarrow{\cong} \widetilde{E}_r\left(A, \{f(DZ_i, y_j)\}\right) \\
\downarrow & \quad \quad \downarrow \\
[A, z \wedge y]^E & \xrightarrow{\cong} [A, f(DZ, y)]^E
\end{align*}
\]
which make this square commute.

The problem of convergence for \( \widetilde{E}_r \) and \( \widetilde{E}_r \) seems somewhat more difficult than for the ordinary Adams spectral sequence. In \([B2]\) it is shown that
\[ \widetilde{E}_r\left(A, \{z_i \wedge s_j\}\right) = [A, z]^E \]
converges if \( \lim_{\xi} z_i = z \), if the complex \( J_\xi \) associated to \( \{z_i\} \) is trivial, and the conditions for convergence of the ordinary Adams spectral sequence
\[ E_r\left(A, z\right) = [A, z]^E \]
are met [Al, §15]. In general the most optimistic conjecture is that
\( \tilde{E}_r(A,[F(X_i,Y_j)]) \) converges to \( [A \land F,Y]^E \) and \( \tilde{E}_r(A,[Z_i \land Y_j]) \) converges
to \( [A,C \land Y]^E \), where

\[ F = \text{Fiber}(X \to \varprojlim X_i) \]
\[ C = \text{Cofiber}(\varprojlim Z_i \to Z) \]

under the conditions which would ensure convergence of the ordinary Adams
spectral sequences abutting to these groups.

To conclude, let us indicate what is new here. The construction of \( \tilde{E}_r \)
was known to Frank Adams in 1968 or 1969, though I learned it from Ravenel's
paper [R1]. The identification of \( \tilde{E}_2 \) in Theorem 5 I learned from Doug
Ravenel. The special case of \( \tilde{E}_r \) in which \( \{Z_i\} \) is the filtration of a
finite complex by skeleta and the associated complex \( J_r \) is trivial was used
by Milgram in [M2]. (He also assumed \( E = HZ_p \).) Another special case of
\( \tilde{E}_r \), in which \( \{Z_i\} \) is an Adams resolution for a different ring theory,
was used by Haynes Miller [M4]. Thus, it is the construction of \( \tilde{E}_r \) in
general and the observation that all these spectral sequences come from two
dual constructions which is new here. This includes all the results about
\( \tilde{E}_r \), essentially 5.6 and 7-10.

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