Characteristic Classes in K-Theory
General Theory

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Outline

1. K-Theory of Classifying Spaces
2. Connective K-Theory
 Representation Rings

 Restriction, induction and conjugation induce natural transformations between the real, complex, and quaternionic representation rings:

\[
\begin{align*}
RO & \quad rc = 2 \\
RU & \quad rc = 1 + \tau \\
\tilde{c} & \quad \tilde{c}q = 1 + \tau \\
RSp & \quad q\tilde{c} = 2
\end{align*}
\]

\[
\begin{align*}
RSp & \quad q\tilde{c} = 2 \\
RU & \quad \tau\tilde{c} = \tilde{c} \\
 \tau & \quad \tau\tilde{c} = \tilde{c} \\
\end{align*}
\]
Representation Rings

For any compact Lie group, we may choose

- irreducible real representations $U_i$,
- irreducible complex representations $V_j$, and
- irreducible quaternionic representations $W_k$

so that

- $RU = \mathbb{Z}\langle cU_i \rangle \oplus \mathbb{Z}\langle V_j, \tau V_j \rangle \oplus \mathbb{Z}\langle \tilde{c}W_k \rangle$
- $RO = \mathbb{Z}\langle U_i \rangle \oplus \mathbb{Z}\langle rV_j \rangle \oplus \mathbb{Z}\langle rqW_k \rangle$
- $RSp = \mathbb{Z}\langle qcU_i \rangle \oplus \mathbb{Z}\langle qV_j \rangle \oplus \mathbb{Z}\langle W_k \rangle$
Equivariant K-theory

Evidently,

\[ KU^0_G = RU(G) \]

and similarly for \( KO \) and \( KSp \). The Atiyah-Segal Theorem asserts that the map \( S \leftarrow EG_+ \) induces completion at the augmentation ideal:

\[ KU^0_G = RU(G) \rightarrow KU^0_G(EG_+) = KU^0(BG) \]

and similarly for \( KO \) and \( KSp \). Thus

\[ KU^0(BG) = RU(G)_\hat{} \quad KO^0(BG) = RO(G)_\hat{} \quad KSp^0(BG) = RSp(G)_\hat{} \]

They also show that \([BG, U] = 0 = [BG, O] = [BG, Sp]\).

Write \( \hat{RO} \) for \( RO_\hat{} \) hereafter.
Using representations of $G$ on Clifford modules, Atiyah, Bott and Shapiro give an elegant account of the Atiyah-Segal isomorphisms, showing:

### Theorem

1. $KU^*_G = RU(G)[\nu, \nu^{-1}]$.
2. $KO^*_G = RO^*(G)[\beta, \beta^{-1}]$ where

<table>
<thead>
<tr>
<th>$RO^i(G)$</th>
<th>$RO^i(G)$/ $RU(G)$</th>
<th>$RO^i(G)$/ $RSp(G)$</th>
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<tbody>
<tr>
<td>$RO^0(G)$</td>
<td>$RO(G)$</td>
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<td>$\mathbb{Z}{U_i, rV_j, r\tilde{c}W_k}$</td>
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<td>$RO^{-1}(G)$</td>
<td>$RO(G)/RU(G)$</td>
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<td>$\mathbb{F}_2{U_i}$</td>
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<td>$RO^{-2}(G)$</td>
<td>$RU(G)/RSp(G)$</td>
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<td>$\mathbb{F}_2{cU_i} \oplus \mathbb{Z}{V_j}$</td>
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<tr>
<td>$RO^{-3}(G)$</td>
<td>$0$</td>
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<td></td>
<td>$\mathbb{Z}{qcU_i, qV_j, W_k}$</td>
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<td>$RO^{-4}(G)$</td>
<td>$RSp(G)$</td>
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<td>$\mathbb{F}_2{W_k}$</td>
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<tr>
<td>$RO^{-5}(G)$</td>
<td>$RSp(G)/RU(G)$</td>
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<td>$\mathbb{Z}{\bar{V}_j} \oplus \mathbb{F}_2{\tilde{c}W_k}$</td>
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<td>$RO^{-6}(G)$</td>
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<tr>
<td>$RO^{-7}(G)$</td>
<td>$0$</td>
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Coefficients

The action of the coefficients,

\[ KU^* = \mathbb{Z}[v, v^{-1}] \]

and

\[ KO^* = \frac{\mathbb{Z}[\eta, \alpha, \beta, \beta^{-1}]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)} \]

with \( v \in KU^{-2}, \eta \in KO^{-1}, \alpha \in KO^{-4}, \) and \( \beta \in KO^{-8}, \) coincide with natural maps in representation theory.
For example, $\eta$ induces the natural quotients

$$RO \rightarrow RO/RU \quad \text{and} \quad RSp \rightarrow RSp/RU$$

and the evident inclusions

$$RO/RU \rightarrow RU/RSp \quad \text{and} \quad RSp/RU \rightarrow RU/RO.$$ 

On the level of Clifford algebras, multiplication by $\eta$ is complexification.
Similarly multiplication by $\alpha$ is quaternionification. Precisely, it is

- $qc : RO \rightarrow RSp$ in degrees 0 mod 8
- $r\tilde{c} : RSp \rightarrow RO$ in degrees 4 mod 8
- multiplication by 2,

$$F_2\{cU_i\} \oplus \mathbb{Z}\{\overline{V}_j\} \rightarrow \mathbb{Z}\{\overline{V}_j\} \oplus F_2\{\tilde{c}W_k\}$$

and

$$\mathbb{Z}\{\overline{V}_j\} \oplus F_2\{\tilde{c}W_k\} \rightarrow F_2\{cU_i\} \oplus \mathbb{Z}\{\overline{V}_j\}$$

in degrees 2 mod 4,

- and 0 in odd degrees.
There is a more elementary deduction of the values of the completed theory, just from the spaces involved in Bott periodicity, together with the isomorphisms [Atiyah?]

\[
[BG, BO \times \mathbb{Z}] = \hat{RO}(G), \quad [BG, BSp \times \mathbb{Z}] = \hat{RSp}(G),
\]

and

\[
[BG, BU \times \mathbb{Z}] = \hat{RU}(G), \quad [BG, U] = 0.
\]
Recall that Bott periodicity says that starting with $BO \times \mathbb{Z}$ and repeatedly taking loops gives

- $O$
- $O/U$
- $U/Sp$
- $BSp \times \mathbb{Z}$
- $Sp$
- $Sp/U$
- $U/O$, and then
- $BO \times \mathbb{Z}$ again.
The values of \([BG, -]\) on the infinite loop spaces above are as follows:

1. \([BG, O] = \hat{RO}(G)/\hat{RU}(G)\)
2. \([BG, Sp] = \hat{RSp}(G)/\hat{RU}(G)\)
3. \([BG, U/Sp] = 0\)
4. \([BG, U/O] = 0\)
5. \([BG, O/U] = \hat{RU}(G)/\hat{RSp}(G)\)
6. \([BG, Sp/U] = \hat{RU}(G)/\hat{RO}(G)\)
Proof.

\([BG, O] = \hat{RO}(G)/\hat{RU}(G)\) by mapping \(BG\) into the fibration sequence

\[
\begin{align*}
\Omega(U) &\longrightarrow \Omega(U/O) \longrightarrow O \longrightarrow U. \\
\| & \quad \| \\
BU \times \mathbb{Z} &\longrightarrow BO \times \mathbb{Z}
\end{align*}
\]

since \([BG, U] = 0\).
Proof.

\([BG, Sp] = \hat{R}Sp(G)/\hat{R}U(G)\) by mapping \(BG\) into the fibration sequence

\[
\begin{array}{c}
\Omega(U) \longrightarrow \Omega(U/Sp) \longrightarrow Sp \longrightarrow U.
\end{array}
\]

\[
\begin{array}{c}
BU \times \mathbb{Z} \longrightarrow BSp \times \mathbb{Z}
\end{array}
\]

since \([BG, U] = 0\).
Proof.

\([BG, U/Sp] = 0\) by mapping \(BG\) into the fibration sequence

\[
U \longrightarrow U/Sp \longrightarrow BSp \times \mathbb{Z} \longrightarrow BU \times \mathbb{Z},
\]

since \(\tilde{c} : \widehat{RSp}(G) \longrightarrow \widehat{RU}(G)\) is a monomorphism.

\([BG, U/O] = 0\) by mapping \(BG\) into the fibration sequence

\[
U \longrightarrow U/O \longrightarrow BO \times \mathbb{Z} \longrightarrow BU \times \mathbb{Z},
\]

since \(c : \widehat{RO}(G) \longrightarrow \widehat{RU}(G)\) is a monomorphism.
Proof.

\[ [BG, O/U] = \hat{RU}(G)/\hat{RSp}(G) \] by mapping \( BG \) into the fibration sequence

\[
\begin{array}{ccccccccc}
\Omega^2(O/U) & \longrightarrow & \Omega(U) & \longrightarrow & \Omega(O) & \longrightarrow & \Omega(O/U) & \longrightarrow & U & \longrightarrow & O.
\end{array}
\]

\[
\begin{array}{ccccccccc}
BSp \times \mathbb{Z} & \longrightarrow & BU \times \mathbb{Z} & \longrightarrow & O/U & \longrightarrow & U/Sp
\end{array}
\]

and using \([BG, U/Sp] = 0\).
Proof.

Finally, for Part (6) we see \([BG, Sp/U] = \widehat{RU}(G)/\widehat{RO}(G)\) by mapping \(BG\) into the fibration sequence

\[
\begin{array}{cccccc}
\Omega^2(Sp/U) & \rightarrow & \Omega(U) & \rightarrow & \Omega(Sp) & \rightarrow & \Omega(Sp/U) \\
\| & & & & & & \\
BO \times \mathbb{Z} & \rightarrow & BU \times \mathbb{Z} & \rightarrow & Sp/U & \rightarrow & U/O \\
\| & & & & & & \\
\end{array}
\]

and using \([BG, U/O] = 0\).
Coefficients

Taking connective covers gives

$$ku^* = \mathbb{Z}[v]$$

and

$$ko^* = \frac{\mathbb{Z}[\eta, \alpha, \beta]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)}.$$

These now relate cohomology and periodic K-theory:

$$\begin{align*}
ko &\xrightarrow{[\beta^{-1}]} KO \\
c &\downarrow \quad c & HZ &\xleftarrow{[\nu^{-1}]} ku &\xrightarrow{[\nu^{-1}]} KU
\end{align*}$$
The cofiber sequence

\[ \Sigma^2 ku \xrightarrow{v} ku \rightarrow H\mathbb{Z} \rightarrow \Sigma^3 ku \]

results in a Bockstein spectral sequence

\[ H\mathbb{Z}^*(X)[v] \Rightarrow ku^*(X) \]

with first differential

\[ Q_1 : H\mathbb{Z} \rightarrow \Sigma^3 H\mathbb{Z} \]

Essentially the same as the Atiyah-Hirzebruch spectral sequence since

\[ \cdots \xrightarrow{v} \Sigma^{2i} ku \xrightarrow{v} \cdots \xrightarrow{v} \Sigma^2 ku \xrightarrow{v} ku \]

is the Postnikov tower of \( ku \).
Similarly, the cofiber sequence

\[ \Sigma \text{ko} \xrightarrow{\eta} \text{ko} \xrightarrow{c} \text{ku} \xrightarrow{R} \Sigma^2 \text{ko} \]

results in a Bockstein spectral sequence

\[ \text{ku}^*(X)[\eta] \implies \text{ko}^*(X) \]

with first differential \( cR \)

- closely related to \( 1 + \tau \) in the periodic theory,
- and to \( Sq^2 \) in cohomology.

Better, since \( \eta^3 = 0 \),

- it collapses: \( E^4 \equiv E^\infty \)
- at \( E^\infty \) it is concentrated on lines 0, 1 and 2.
The $\eta$-$c$-$R$ sequence, its differential, and related operations
Since $H^* ku = A/E(1)$ and $H^* ko = A/A(1)$, we have Adams spectral sequences

\[
\text{Ext}^{s,t}_{E(1)}(F_2, H^* BG) \Rightarrow ku^{-t+s} BG
\]

and

\[
\text{Ext}^{s,t}_{A(1)}(F_2, H^* BG) \Rightarrow ko^{-t+s} BG
\]

(Associated graded cannot distinguish $ku^* BG$ from $ku^*_G$.) 'Accounting device', destroys multiplicative information.

Typical use: show the Bott map acts monomorphically in a range beyond the edge of periodicity, so that relations can be accurately detected in periodic K-theory, which is determined by representation theory. Use the Adams spectral sequence to verify that the implications of these relations suffice.
Since $H^*ko$, $H^*HZ$ and $H^*H$ are all induced up from $A(1)$, we can compute the cohomology of the Postnikov sections of $ko$ in $A(1)$-Mod and then tensor up to $A$.

\[
\begin{array}{ccccccc}
  & & & & & ko & \rightarrow & HZ & \rightarrow & \Sigma ko\langle 1 \rangle \\
 0 & \rightarrow & & & & & & & & \\
 1 & \rightarrow & & & & & & & & \\
 2 & \rightarrow & & & & & & & & \\
 3 & \rightarrow & & & & & & & & \\
 4 & \rightarrow & & & & & & & & \\
 5 & \rightarrow & & & & & & & & \\
\end{array}
\]
$ko\langle 1 \rangle \rightarrow H \rightarrow \Sigma ko\langle 2 \rangle$
Connective K-Theory

Postnikov Tower of $ko$

$ko\langle 2 \rangle \rightarrow \Sigma^2 H \rightarrow \Sigma ko\langle 4 \rangle$

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$ko\langle 4 \rangle \rightarrow \Sigma^4 HZ \rightarrow \Sigma ko\langle 8 \rangle$
Corollary

The primary differentials in the Atiyah-Hirzebruch spectral sequences $H^p(X, ko_q) \Longrightarrow ko^{p-q}(X)$ and $H^p(X, KO_q) \Longrightarrow KO^{p-q}(X)$ are:

- $H^p(X, ko_8i) \xrightarrow{d_2} H^{p+2}(X, ko_8i+1)$ is $H^p(X, \mathbb{Z}) \xrightarrow{Sq^2} H^{p+2}(X, \mathbb{Z}/2)$,
- $H^p(X, ko_8i+1) \xrightarrow{d_2} H^{p+2}(X, ko_8i+2)$ is $H^p(X, \mathbb{Z}/2) \xrightarrow{Sq^2} H^{p+2}(X, \mathbb{Z}/2)$,
- $H^p(X, ko_8i+2) \xrightarrow{d_3} H^{p+3}(X, ko_8i+4)$ is $H^p(X, \mathbb{Z}/2) \xrightarrow{Sq^3} H^{p+3}(X, \mathbb{Z})$, and
- $H^p(X, ko_8i+4) \xrightarrow{d_5} H^{p+5}(X, ko_8i+8)$ is $H^p(X, \mathbb{Z}) \xrightarrow{Sq^5} H^{p+5}(X, \mathbb{Z})$. 
The crude truncation $ku = KU[0, \infty)$ which we used in the non-equivariant case will not produce an interesting result in the equivariant case. In particular it will not have Euler classes, and would not be complex orientable.

Solution: observe that any equivariant $ku_G$ should sit in a commutative square

$$
\begin{array}{c}
\text{ku}_G \\
\downarrow \\
F(EG_+, ku_G)
\end{array}
\xymatrix{
\text{ku}_G \\
\downarrow \\
\approx \\
\downarrow \\
F(EG_+, \inf^G ku) \\
\downarrow \\
F(EG_+, \inf^G KU)
}

and define $ku_G$ to be the pullback.
Greenlees [JPAA 2004] showed this has good properties:

1. $\text{ku}_G$ is a (strict) commutative ring $G$-spectrum.
2. If $H \subset G$ then $\text{res}^G_H \text{ku}_G = \text{ku}_H$.
3. $\text{ku}_G$ is a split ring $G$-spectrum.
4. $\text{ku}_G[v^{-1}] = \text{KU}_G$.
5. $\text{ku}_G^*$ is Noetherian.
6. $\text{ku}_G$ is complex orientable.
7. $\text{ku}_G^* \rightarrow \text{ku}^* BG$ is completion
8. There is a local cohomology spectral sequence.
The same construction works in the real case. We define $ko_G$ to be the pullback.

$$
\begin{array}{ccc}
ko_G & \longrightarrow & KO_G \\
\downarrow & & \downarrow \\
F(EG_+, \inf_1^G ko) & \longrightarrow & F(EG_+, \inf_1^G KO)
\end{array}
$$
Calculational consequence

The coefficient rings sit in pullback squares

\[
\begin{align*}
ku_G^* & \rightarrow KU_G^* \\
\downarrow & \downarrow \\
kku(BG) & \rightarrow KU(BG)
\end{align*}
\]

\[
\begin{align*}
ko_G^* & \rightarrow KO_G^* \\
\downarrow & \downarrow \\
ko(BG) & \rightarrow KO(BG)
\end{align*}
\]
More interesting, if \( \widehat{G} = G \times C_2 \), there is a \( \widehat{G} \) spectrum \( KR \) representing \( G \)-equivariant periodic Real K-theory, in the sense of Atiyah. There exists a \( C_2 \)-map \( KR \rightarrow KR^G \) which is a \( C_2 \)-equivalence, hence a \( \widehat{G} \)-map \( \text{inf}_{C_2}^G KR \rightarrow KR \) of \( \widehat{G} \)-spectra which is a \( C_2 \)-equivalence, so that

\[
F(EG_+, KR) \simeq F(EG_+ , \text{inf}_{C_2}^G KR)
\]

as \( \widehat{G} \)-spectra.
We then define $kR_G$ to be the pullback in $\hat{G}$-spectra

\[
\begin{array}{ccc}
kR_G & \to & KR \\
\downarrow & & \downarrow \\
F(EG_+, \inf_{C_2} \hat{G} kR) & \to & F(EG_+, \inf_{C_2} \hat{G} KR)
\end{array}
\]

where the lower left $kR$ is the connective cover of the $C_2$-spectrum $KR$.

**Theorem**

As $G$-spectra, $kR_G \simeq ku_G$ and $(kR_G)^{C_2} \simeq ko_G$. 
There are compatible Chern classes in cohomology, in connective and periodic $K$-theories, and in representation theory

\[
\begin{array}{cccccc}
\chi_i^H & \leftarrow & \chi_i^{ku} & \overset{v^i}{\rightarrow} & \chi_i^K & \leftarrow & \chi_i^R \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{2i}(BU(n); \mathbb{Z}) & \leftarrow & ku^{2i} BU(n) & \overset{v^i}{\rightarrow} & K^0 BU(n) & \leftarrow & R(U(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{2i}(BT^n; \mathbb{Z}) & \leftarrow & ku^{2i} BT^n & \overset{v^i}{\rightarrow} & K^0 BT^n & \leftarrow & R(T^n) \\
\end{array}
\]

which restrict to the symmetric polynomials in the Euler classes of the natural line bundles on a maximal torus $T^n \subset U(n)$. 
Chern classes in representation theory

Definition
Let \( n = \dim(V) \). Then

\[
c_k^R(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V)
\]

Definition
The Chern (or 'gamma') filtration of representation theory: \( JU_i(G) \subset R(G) \) is the ideal generated by all products \( c_{i_1}(V_1) \cdots c_{i_k}(V_k) \) with \( i_1 + \cdots + i_k \geq i \).

This is multiplicative:

\[
JU_{j}(G)JU_{k}(G) \subset JU_{j+k}(G)
\]
The formula for the $c_k$ in terms of the $\Lambda^i$ is the same as the formula for the $\Lambda^k$ in terms of the $c_i$

\[
\Lambda^k(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} c_i^R(V)
\]

since this formula is the one that relates symmetric polynomials in variables $t_1, \ldots, t_n$ to symmetric polynomials in $1 - t_1, \ldots, 1 - t_n$, and $t \mapsto 1 - t$ is an involution.
Modified Rees ring

Given a ring $R$ and a multiplicative filtration $\mathcal{F} = \{ R = F_0 \supset F_1 \supset \cdots \}$ the Modified Rees ring

$$M\text{Rees}(R, \mathcal{F}) = \left\{ \sum_{i=-N}^{\infty} r_i t^i \mid r_{-i} \in F_i \text{ for } i > 0 \right\}$$

The usual Rees ring construction uses $F_i = I_i$ for an ideal $I \subset R$.

If $\mathcal{C}$ is the Chern filtration of $R(G)$ then

$$M\text{Rees}(RU(G)) := M\text{Rees}(RU(G), \mathcal{C})$$

is a very good approximation to $ku_G^*$. 
The Complex Case

Lemma

\( JU_1(G) \) is the augmentation ideal \( JU(G) \), consisting of representations of virtual dimension 0.

Proof.

Since \( JU \) is generated by first Chern classes, \( c_1^R(V) = \dim(V) - V \),

\[ JU \subset JU_1. \]
Proof.

(Cont.) Conversely, for $k > 0$,

$$\dim(c_k^R(V)) = \dim \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V)$$

$$= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \dim \Lambda^i(V)$$

$$= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \binom{n}{i}$$

$$= \binom{n}{n-k} \sum_{i=0}^{k} (-1)^i \binom{k}{i}$$

$$= 0$$

so $J U_1 \subset J U$.  \[ \square \]
Lemma

$JU_2$ consists of representations of virtual dimension 0 and virtual determinant 1.

For the defining representation of $U(n)$,

$$\det = \Lambda^n = 1 - c_1^R + \cdots + (-1)^n c_n^R$$

$$= 1 - \nu c_1^{ku} + \cdots + (-\nu)^n c_n^{ku}.$$

Since $SU(n)$ is the fiber of $\det : U(n) \to U(1)$,

$$ku^* BSU(n) = ku^* BU(n)/(c_1^{ku}(\det))$$

$$= ku^* [[c_1, \ldots, c_n]]/((1 - \Lambda^n)/\nu)$$

$$= ku^* [[c_1, \ldots, c_n]]/(c_1 - \nu c_2 + \nu^2 c_3 - \cdots + (-\nu)^{n-1} c_n).$$

('Unnaturally' isomorphic to $ku^* [[c_2, \ldots, c_n]]$.)
$JU_2$ is generated by products of Chern classes and by $c_i(V)$ for $i \geq 2$.

Let $JU'_2$ be the ideal of virtual dimension 0, determinant 1 representations, i.e., those whose classifying map lifts over $BSU \to BU \to BU \times \mathbb{Z}$.

For $V - W \in JU'_2$, let

$$\delta = \det(V) = \det(W) \quad \text{and} \quad n = \dim(V) = \dim(W).$$

$\delta \in R(G)^\times$, so it suffices to show $V\delta^{-1} - W\delta^{-1} \in JU_2$.

So we may assume $\delta = 1$. 

Since $V - W = (n - W) - (n - V)$, it suffices to show
\[ c_1(W) = n - W \in JU_2. \]

\[ \det(W) = 1 \] so the representation $W$ lifts to $SU(n)$.

In $ku^* BSU(n)$,
\[ c_1 = vc_2 - v^2 c_3 + \cdots \pm v^{n-1} c_n \]
and it follows that $JU'_2 \subset JU_2$. 
Conversely, we must show that any Chern class \( c_k(V) \), \( k > 1 \), and any product \( c_1(V)c_1(W) \), have dimension 0 and determinant 1. The first was shown already. For the second, recall that

\[
\det(kV) = (\det(V))^k \quad \text{and} \quad \det(\Lambda^i(V)) = (\det(V))^{(n-1)}(i-1)
\]

for \( i > 0 \). Thus

\[
\det(c_k(V)) = \prod_{i=1}^{k} (\det(\Lambda^i(V)))^{(-1)^i(n-i)}(n-k)
\]

\[
= \prod_{i=1}^{k} (\det(V))^{(-1)^i(n-i)(n-1)}(n-k)(n-1)
\]

This is \( \det(V) \) raised to the power

\[
\sum_{i=1}^{k} (-1)^i\binom{n-i}{n-k}\binom{n-1}{i-1} = \binom{n-1}{k-1} \sum_{i=1}^{k} (-1)^i\binom{k-1}{i-1} = 0,
\]

since \( k - 1 > 0 \).
Finally, if \( m = \dim(V) \) and \( n = \dim(W) \). Then

\[
\dim(c_1(V)c_1(W)) = \dim((m - V)(n - W)) = 0
\]

and

\[
\det(m - V)(n - W)) = \det(mn - nV - mW + VW) = 1
\]

since

\[
\det(VW) = (\det(V))^n(\det(W))^m.
\]
Theorem

\[ ku_G^* \rightarrow KU_G^* \text{ is a monomorphism in codegrees } \leq 5 \text{ and } \]

\[ ku_G^i = \begin{cases} 
0 & i \leq 0 \text{ odd} \\
RU(G) & i \leq 0 \text{ even} \\
0 & i = 1 \\
JU(G) & i = 2 \\
0 & i = 3 \\
JU_2(G) & i = 4 \\
0 & i = 5 
\end{cases} \]

Proof.

Compare the Atiyah-Hirzebruch spectral sequences

\[ H^p(BG, ku_q) \longrightarrow ku^{p-q}(BG) \]

\[ H^p(BG, KU_q) \longrightarrow KU^{p-q}(BG) \]
Proof.

(Cont.) The only differential which could affect the difference between them is

\[ d_3 : H^2(BG, KU_{-2}) \longrightarrow H^5(BG, KU_0). \]

But every element of \( H^2(BG, KU_{-2}) \) is a first Chern class, and these survive by the universal example.

This gives \( ku^5BG = 0 \) and shows that \( ku^4BG \) is the kernel of the map \( KU^4_G \longrightarrow H^2(BG; \mathbb{Z}) \otimes H^0(BG, \mathbb{Z}) \) induced by the Postnikov section

\[ BU \times \mathbb{Z} \xrightarrow{B \text{det} \times 1} BU(1) \times \mathbb{Z} \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 0). \]

By the Lemma, this is exactly \( \widehat{JU}_2(G) \). The pullback diagram then gives the uncompleted results.
Beyond this, complications set in.

**Theorem**

There are exact sequences

\[ 0 \to H^3 BG \xrightarrow{\overline{Q}_1} ku^6 BG \xrightarrow{\nu} ku^4 BG = \widehat{JU}_2(G) \to H^4 BG \xrightarrow{\overline{Q}_1} ku^7 BG \to 0 \]

and

\[ 0 \to H^5 BG \xrightarrow{\overline{Q}_1} ku^8 BG \xrightarrow{\nu} ku^6 BG \to H^6 BG \xrightarrow{\overline{Q}_1} ku^9 BG \to \ldots \]

**Proof.**

Use \( \Sigma^2 ku \xrightarrow{\nu} ku \to H\mathbb{Z} \).
The Real Case

**Theorem**

$k_0_G^* \rightarrow KO_G^*$ is a monomorphism in codegrees $\leq 7$, and

$$
\begin{align*}
\text{ko}_G^i & = \\
0 & \quad i = 1 \\
JU(G)/JO(G) & \subset RU(G)/RO(G) \quad i = 2 \\
JSp(G)/JU(G) & \subset RSp(G)/RU(G) \quad i = 3 \\
JSp(G) & \subset RSp(G) \quad i = 4 \\
0 & \quad i = 5 \\
JU_2(G)/JSp(G) & \subset RU(G)/RSp(G) \quad i = 6 \\
JSpin(G)/JU_2(G) & \subset RO(G)/RU(G) \quad i = 7
\end{align*}
$$
As in the complex case $\text{ko}^i_G \to \text{KO}^i_G$ is often a monomorphism for $i = 8$ or 9. There is an exact sequence

$$J\text{SO} \xrightarrow{\beta w_2} H^3(BG; \mathbb{Z}) \to \text{ko}^8_G \to \text{KO}^{-8}_{-8} = RO(G)$$

and $\text{ko}^9_G = 0$ iff the two maps

$$J\text{SO} \xrightarrow{w_2} H^2(BG; \mathbb{Z}/2) \text{ and } R\text{Spin}(G) \xrightarrow{p_1/2} H^4(BG; \mathbb{Z})$$

are monomorphisms.
The argument goes as in the real case, but the difference between the Atiyah-Hirzebruch spectral sequences for $ko^*BG$ and $KO^*BG$ is more complicated. The analysis is helped by the following.

**Theorem**

*The first nine spaces in the spectrum $ko$ and the Moore-Postnikov factorization of $ko \rightarrow KO$ are*

1. $BO \times \mathbb{Z} \rightarrow BO \times \mathbb{Z}$
2. $U/O \rightarrow U/O$
3. $Sp/U \rightarrow Sp/U$
4. $Sp \rightarrow Sp$
5. $BSp \rightarrow BSp \times \mathbb{Z}$
6. $SU/Sp \rightarrow U/Sp$
7. $Spin/SU \rightarrow SO/U \rightarrow O/U$
8. $String \rightarrow Spin \rightarrow SO \rightarrow O$
9. $BString \rightarrow BSpin \rightarrow BSO \rightarrow BO \rightarrow BO \times \mathbb{Z}$
This is proved at the same time as we identify maps from $BG$ into various of these homogeneous spaces. Denote the connective covers of $O$ by $SO = O\langle 1 \rangle$, $Spin = O\langle 3 \rangle$, and $String = O\langle 7 \rangle$. Then

**Theorem**

For the 0, 2, and 6-connected covers of $O$ and associated homogeneous spaces, we have:

1. $\Omega(SO/U) = U/Sp$ and $[BG, SO/U] = \hat{J}U(G)/\hat{J}Sp(G)$
2. $\Omega(Spin/SU) = SU/Sp$ and $[BG, Spin/SU] = \hat{J}U_2(G)/\hat{J}Sp(G)$
3. $\Omega SO = O/U$ and $[BG, SO] = \hat{J}O(G)/\hat{J}U(G)$
4. $\Omega Spin = SO/U$ and $[BG, Spin] = \hat{J}SO(G)/\hat{J}U(G)$
5. $\Omega String = Spin/SU$ and $[BG, String] = \hat{J}Spin(G)/\hat{J}U_2(G)$
The most interesting of these is $String$. The relevant diagram in this case is

\[
\begin{array}{c}
\Sigma^2 \mathbb{H} \mathbb{Z} \rightarrow \Sigma^2 \mathbb{H} \mathbb{Z}/2 \rightarrow \Sigma^3 \mathbb{H} \mathbb{Z} \\
BU \rightarrow BSO \rightarrow Spin \\
BSU \rightarrow BSpin \rightarrow String
\end{array}
\]

The latter fibre sequence shows that $\Omega(String) = Spin/SU$ and that $[BG, String]$ is $\widehat{\text{JSpin}}(G)/\widehat{\text{JU}}_2(G)$, since $[BG, B^2 SU] = [BG, U(5)] = 0$. 
To get the fiber sequence in the middle row of the previous diagram, consider:

\[
\begin{array}{c}
\Sigma \mathbb{H}_2 / 2 \quad \Sigma \mathbb{H}_2 / 2 \\
\uparrow w_1 \quad \uparrow \Omega w_2 \\
BU \quad BO \quad SO \\
\| \quad \| \quad \| \\
BU \quad BSO \quad Spin
\end{array}
\]

The latter fibre sequence shows that \([BG, Spin] = \widehat{JSO}(G)/\widehat{JU}(G)\) since, again, \([BG, B^2U] = [BG, SU] = 0\).
End of Part One