HIRSCH’S FORMULA FOR SYMMETRIC MASSEY PRODUCTS

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We will work mod 2 for simplicity. The Hirsch formula says that cup-1 with a fixed class is a derivation:
\[(xy) \cup_1 z = x(y \cup_1 z) + (x \cup_1 z)y\]
All good cup-1 products have this property.

Lemma 1. If the Hirsch formula holds, then the Massey product
\[(x \cup_1 x)y \in \langle x, y, x \rangle\]

Proof: Suppose that \(d(a) = xy\). Since
\[d(y \cup_1 x) = xy + yx\]
we have
\[d(a + y \cup_1 x) = yx,\]
and hence the Massey product
\[ax + xa + x(y \cup_1 x) \in \langle x, y, x \rangle.\]
Now
\[d(a \cup_1 x) = ax + xa + (xy) \cup_1 x,\]
so that
\[(xy) \cup_1 x + x(y \cup_1 x) \in \langle x, y, x \rangle\]
in homology. But the Hirsch formula says that
\[(xy) \cup_1 x = x(y \cup_1 x) + (x \cup_1 x)y\]
so we have
\[(x \cup_1 x)y \in \langle x, y, x \rangle\]

References