HIRSCH'S FORMULA FOR SYMMETRIC MASSEY PRODUCTS

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We will work mod 2 for simplicity. The Hirsch formula says that cup-1 with a fixed class is a derivation:

$$(xy) \cup_1 z = x(y \cup_1 z) + (x \cup_1 z)y$$

All good cup-1 products have this property.

Lemma 1. *If the Hirsch formula holds, then the Massey product*

$$(x \cup_1 x)y \in \langle x, y, x\rangle$$

Proof: Suppose that $d(a) = xy$. Since

$$d(y \cup_1 x) = xy + yx$$

we have

$$d(a + y \cup_1 x) = yx,$$

and hence the Massey product

$$ax + xa + x(y \cup_1 x) \in \langle x, y, x\rangle.$$

Now

$$d(a \cup_1 x) = ax + xa + (xy) \cup_1 x,$$

so that

$$(xy) \cup_1 x + x(y \cup_1 x) \in \langle x, y, x\rangle$$

in homology. But the Hirsch formula says that

$$(xy) \cup_1 x = x(y \cup_1 x) + (x \cup_1 x)y$$

so we have

$$(x \cup_1 x)y \in \langle x, y, x\rangle$$

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References

[1] Guy Hirsch, Quelques propriétés des produits de Steenrod. (French) C. R. Acad. Sci. Paris **241** (1955), 923 – 925. MR0073182 (17,396c).