ON THE POSTNIKOV TOWERS FOR REAL AND COMPLEX CONNECTIVE K-THEORY

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1. Introduction

The analysis of real connective K-theory is facilitated by the \( \eta cR \) cofiber sequence

\[
\Sigma k_o \underset{\eta}{\rightarrow} k_o \overset{c}{\rightarrow} k_u \overset{R}{\rightarrow} \Sigma^2 k_o
\]

relating real and complex K-theories [2]. Here we extend this relationship through the Postnikov towers, producing several useful \( ko \)-module maps in the process.

Theorem 1. The \( \eta cR \) sequence lifts to cofiber sequences relating the connective covers of \( ko \) and \( ku \) as follows:

\[
\begin{align*}
\Sigma k_o & \overset{\eta}{\rightarrow} k_o \overset{c}{\rightarrow} k_u \overset{R}{\rightarrow} \Sigma^2 k_o \\
\Sigma k_o & \overset{\eta_1}{\rightarrow} k_o(1) \overset{c_1}{\rightarrow} \Sigma k_u \overset{r_r}{\rightarrow} \Sigma^2 k_o \\
\Sigma k_o(1) & \overset{\eta_2}{\rightarrow} k_o(2) \overset{c_2}{\rightarrow} \Sigma^2 k_u \overset{r_r}{\rightarrow} \Sigma^2 k_o(1) \\
\Sigma k_o(2) & \overset{\eta_4}{\rightarrow} k_o(4) \overset{c_4}{\rightarrow} \Sigma^2 k_u \overset{r_r}{\rightarrow} \Sigma^2 k_o(2) \\
\Sigma k_o(4) & \overset{\eta_8}{\rightarrow} k_o(8) \overset{c_8}{\rightarrow} \Sigma^2 k_u \overset{r_r}{\rightarrow} \Sigma^2 k_o(4) \\
\Sigma k_o(8) & \overset{\eta_8}{\rightarrow} k_o(8) \overset{\Sigma^8 c}{\rightarrow} \Sigma^8 k_u \overset{\Sigma^8 R}{\rightarrow} \Sigma^8 k_o(8)
\end{align*}
\]

In the sequence above, \( c \) is complexification, \( r \) is realification, and \( \eta \) is multiplication by \( \eta \in k_o \). The map \( R \) is an extension of realification \( r \) over the Bott map: \( r = Rv \).

We will write \( X \langle n \rangle \rightarrow X \) for the \( n \)-connected cover of \( X \). By this we mean that \( \pi_i X \langle n \rangle = 0 \) for \( i < n \), while \( \pi_i X \langle n \rangle \rightarrow \pi_i X \) is an isomorphism for \( i \geq n \). It will be useful to record the maps induced in cohomology. All the modules and maps we will deal with are in the image of induction from \( A(1) \)-Mod,

\[ A \otimes_{A(1)} - : A(1) \text{-Mod} \rightarrow A \text{-Mod}, \]

so we will record the results in \( A(1) \)-Mod, leaving it to the reader to tensor up.

The first lift, \( \eta_1 c_1 r \), was brought to my attention by Vic Snaith ([3]). The remaining lifts appeared at one point to be useful in Geoffrey Powell’s analysis of \( k_o^* BV_+ \) ([4]), but in the end were unnecessary there.

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2. Complex Periodicity

In the complex case, periodicity and the Postnikov tower amount to the same thing. If we write \( ku = \mathbb{Z}[v] \), with \(|v| = 2\), then the Postnikov covers of \( ku \) are simply given by multiplication by powers of \( v \).

\[
\begin{array}{c}
\Sigma^2 ku \\
\simeq \\
k u(2) \longrightarrow k u
\end{array}
\quad \text{and more generally}
\begin{array}{c}
\Sigma^{2i+2} ku \\
\simeq \\
k u(2i + 2) \longrightarrow k u(2i)
\end{array}
\]

**Proposition 2.** \( ku \longrightarrow H \mathbb{Z} \longrightarrow \Sigma^3 ku \) induces the short exact sequence

\[
\begin{array}{c}
A(1)/(Sq^1, Sq^3) \quad A(1)/(Sq^1) \quad \Sigma^3 A(1)/(Sq^1, Sq^3)
\end{array}
\]

3. Real Periodicity

In the real case, periodicity is broken into 4 steps. We write \( ko_* = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta) \) with \(|\eta| = 1, |\alpha| = 4, \) and \(|\beta| = 8\).
The following Proposition is well known. It is a simple way to show that a spectrum whose cohomology is $A/\langle A(1) \rangle$ must have 2-local homotopy additively isomorphic to $\pi_* ko$.

**Proposition 3.** The maps induced in cohomology by the Postnikov tower for $ko$ are as follows.

(1)

\[
\begin{array}{c}
ko \\
\rightarrow \\
H_\mathbb{Z} \\
\rightarrow \\
\Sigma ko(1)
\end{array}
\]

induces the short exact sequence

\[
\begin{array}{c}
F_2 \\
\leftarrow \\
\longrightarrow \\
\longrightarrow \\
\Sigma A(1)/(Sq^1) \\
\leftarrow \\
\longrightarrow \\
\Sigma^2 A(1)/(Sq^2)
\end{array}
\]

(2)

\[
\begin{array}{c}
ko(1) \\
\rightarrow \\
\Sigma HF_2 \\
\rightarrow \\
\Sigma ko(2)
\end{array}
\]

induces the short exact sequence

\[
\begin{array}{c}
\Sigma A(1)/(Sq^2) \\
\leftarrow \\
\longrightarrow \\
\longrightarrow \\
\Sigma A(1) \\
\leftarrow \\
\longrightarrow \\
\Sigma (Sq^2) \cong \Sigma^3 A(1)/(Sq^3)
\end{array}
\]
induces the short exact sequence
$\Sigma^2 A(1)/\langle Sq^3 \rangle \leftarrow \Sigma^2 A(1) \leftarrow \Sigma^2 (Sq^3) \cong \Sigma^5 A(1)/(Sq^1, Sq^2 Sq^3)$

induces the short exact sequence
$\Sigma^4 A(1)/(Sq^1, Sq^2 Sq^3) \leftarrow A(1)/(Sq^1) \leftarrow \Sigma^9 F_2$

4. Maps of Postnikov towers

First, we record the maps induced in cohomology by our starting point, the $\eta_c R$ sequence.

**Proposition 4.** $ko \xrightarrow{L} ku \xrightarrow{R} \Sigma^2 ko$ induces the short exact sequence
$A(1)/(Sq^1, Sq^2) \leftarrow A(1)/(Sq^1, Sq^3) \leftarrow \Sigma^2 A(1)/(Sq^1, Sq^2)$
We will now prove Theorem 1 in a series of steps. We start with the braid of cofibrations induced by the composite \( ko \xrightarrow{c} ku \xrightarrow{} HZ \).

\[ \begin{array}{c}
\Sigma ko \\
\eta \\
\eta_1 \\
k_0(1) \\
\Sigma^2 ku \\
\Sigma ko \langle 1 \rangle \\
c_1 \\
ku \\
\Sigma ko \langle 2 \rangle \\
c_2 \\
\Sigma^4 ku \\
\Sigma ko \langle 1 \rangle \langle 1 \rangle \\
\end{array} \]

This gives the \( \eta_1 c_1 r \) sequence. To continue to the next step, we will need to know the maps induced in cohomology by this one.

**Proposition 5.** \( \Sigma ko \xrightarrow{\eta_1} ko(1) \xrightarrow{c_1} \Sigma^2 ku \) induces the short exact sequence

\[ \Sigma F_2 \xrightarrow{\Sigma A(1)/Sq^2} \Sigma^2 A(1)/(Sq^1, Sq^3) \]

**Proof.** These are the only maps which can make the long exact sequence exact. \( \square \)

From this we observe that we have a commutative square

\[ \begin{array}{c}
\Sigma ko \\
\eta \\
k_0(1) \\
\Sigma^2 ku \\
\Sigma ko \langle 1 \rangle \\
\end{array} \]

which induces the following map of cofiber sequences. The map induced in cohomology by \( \eta_1 \) implies that the left hand map \( \Sigma ko \longrightarrow \Sigma HZ \) is nontrivial. This implies that the fiber of \( c_2 \) is \( \Sigma ko(1) \), giving the next Postnikov lift of the \( \eta c R \) sequence.

\[ \begin{array}{c}
\Sigma HZ \\
\Sigma ko \\
\eta_1 \\
k_0(1) \\
\Sigma ko \langle 2 \rangle \\
\end{array} \]

Again we need to record the maps induced in cohomology for use in the next step.
Proposition 6. \( \Sigma ko(1) \xrightarrow{\eta_2} ko(2) \xrightarrow{c_2} \Sigma^4 ku \) induces the short exact sequence
\[
\Sigma^2 A(1)/(Sq^2) \xleftarrow{\eta_2} \Sigma^2 A(1)/(Sq^3) \xleftarrow{c_2} \Sigma^4 A(1)/(Sq^1, Sq^3)
\]

Now consider the braid of cofibrations induced by the composite \( ko(4) \to ko(2) \to \Sigma^4 ku \).

Since \( \eta_2^* \) is nonzero in degree 2, the map \( \Sigma ko(1) \to \Sigma^2 HF_2 \) is nontrivial, and hence the fiber of \( c_4 \) is \( \Sigma ko(2) \). Again, we need to record the maps induced in cohomology, and again, they ‘roll’ one step to the left.

Proposition 7. \( \Sigma^3 ku \xrightarrow{r_2} \Sigma ko(2) \xrightarrow{\eta_4} ko(4) \) induces the short exact sequence
\[
\Sigma^3 A(1)/(Sq^1, Sq^3) \xleftarrow{r_2} \Sigma^3 A(1)/(Sq^3) \xleftarrow{\eta_4} \Sigma^4 A(1)/(Sq^1, Sq^2 Sq^3)
\]

Since \( \eta_4^* \) sends the generator to \( Sq^1 \), we get a map of cofiber sequences whose fiber gives the next lift, \( \eta_8 c_8 r_4 \).
**Proposition 8.** $\Sigma^5 ku \xrightarrow{r_4} \Sigma ko(4) \xrightarrow{\eta_8} ko(8)$ induces the short exact sequence

\[
\Sigma^5 \mathcal{A}(1)/(Sq^1, Sq^3) \xrightarrow{r_4} \Sigma^5 \mathcal{A}(1)/(Sq^1, Sq^2 Sq^3) \xleftarrow{\eta_8} \Sigma^8 F_2
\]

Finally, consider the braid of cofibrations induced by the composite $\Sigma ko(8) \to \Sigma ko(4) \xrightarrow{\eta_8} ko(8)$.

Since $r_4^*$ is an isomorphism on $H^5$, the map $\Sigma^5 ku \to \Sigma^5 HZ$ must be the bottom cohomology generator, justifying the appearance of $\Sigma^8 ku$ and $v$ in this braid.

The result is a cofiber sequence $\Sigma^9 ko \to \Sigma^8 ko \to \Sigma^8 ku$. The maps are $ko$-module maps by construction, and agree with the 8-fold suspensions of $\eta, c$ and $R$ in homotopy, by the maps $X(8) \to X$. The adjunction $F_{ko}(\Sigma^9 ko, \Sigma^8 ko) \simeq F(S^9, \Sigma^8 ko)$, shows that a $ko$-module map $\Sigma^9 ko \to \Sigma^8 ko$ is determined by its effect on homotopy. Therefore, the first map, and hence the other two, are the 8-fold suspensions of $\eta, c$ and $R$. $\square$

**References**


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