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AN EXAMPLE IN THE COHOMOLOGY OF AUGMENTED ALGEBRAS

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We show that the cohomology of augmented algebras is quite sensitive to changes in the augmentation by exhibiting algebras $\bar{A}_0$ and $\bar{A}_1$, isomorphic as algebras but not as augmented algebras, such that $H^*\bar{A}_1$ is commutative but $H^*\bar{A}_0$ is not.

We shall work exclusively over the field $\mathbb{Z}_2$ of 2 elements, though there are odd primary analogs of $\bar{A}_1$ and $\bar{A}_0$ whose cohomologies are similarly related.

We will need Priddy's results on the cohomology of Koszul algebras [3], so we begin by summarizing these. Let $T\{x_i\}$ be the tensor algebra generated by the set $\{x_i\}$, with augmentation $\varepsilon(x_i)=0$. An augmented algebra A is a *pre-Koszul algebra* if there is an epimorphism of augmented algebras $\alpha : T\{x_i\} \to A$ whose kernel is the two-sided ideal generated by elements of the form $\sum g_i x_i + \sum f_{ij} x_i x_j$, where g_i and f_{ij} are in $\mathbb{Z}_2$. Clearly we may assume the $a_i = \alpha(x_i)$ are linearly independent, in which case we call α a *Koszul presentation* and $\{a_i\}$ a set of *Koszul generators*. The pre-Koszul algebra A is homogeneous if all the g_i can be taken to be 0. A *homogeneous Koszul algebra* is a homogeneous pre-Koszul algebra A such that H^*A is generated as an algebra by the cocycles dual to the a_i. In practice, this is verified by showing that A has a basis of monomials in the a_i of particularly nice form [3, Theorem 5.3]. The algebra $T\{x_i\}$ is filtered by letting $F_pT\{x_i\}$ be spanned by all monomials of length p or less. If α is a Koszul presentation of the pre-Koszul algebra A, let $F_pA = \alpha(F_pT\{x_i\})$. The associated graded algebra E^0A is a homogeneous pre-Koszul algebra with Koszul generators b_i, the images of the a_i, and relations $\sum f_{ij} b_i b_j$. Note that A is homogeneous iff $A \cong E^0A$. A *Koszul algebra* is a pre-Koszul algebra A such that E^0A is a homogeneous Koszul algebra.

To state the main theorem of [3], let A be a Koszul algebra with Koszul generators $\{a_i \mid i \in J\}$. Let B be a $\mathbb{Z}_2$ basis of A containing 1, the a_i, and certain monomials $a_{i_1}a_{i_2}\dots a_{i_n}$. Let $S \subset \bigcup_{n\geq 1} J^n$ be such that for each $a \in B-\{1\}$ there is a unique $(i_1, i_2, \dots, i_n) \in S$ with $a = a_{i_1}a_{i_2}\dots a_{i_n}$. The relations for A may be written

$$a_i a_j = \sum_k g_{ijk} a_k + \sum_{(k,l)\in S} f_{ijkl} a_k a_l,$$

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and the relations for E^0A will then be

$$b_i b_j = \sum_{(k,l)\in S} f_{ijkl} b_k b_l.$$

Let $\beta_i \in H^*E^0A$ be the cocycle dual to b_i.

Theorem (Priddy [3, Theorem 4.6]). *H^*E^0A is the algebra generated by $\{\beta_i\}$ with relations*

$$\beta_k\beta_l = \sum_{(i,j)\notin S} f_{ijkl}\beta_i\beta_j$$

for each $(k,l)\in S$. Further, $H^*A = H(H^*E^0A,\delta)$ where

$$\delta\beta_k = \sum_{(i,j)\notin S} g_{ijk}\beta_i\beta_j. \quad \square$$

One technical point must be addressed to justify our intended application of the preceding result. In [3], it is assumed that A is of finite type. However, it is sufficient to assume that H_*A is of finite type, since the results of [3] hold in homology without finite type assumptions, and if H_*A has finite type, there is no problem in dualizing to obtain H^*A.

We are now prepared to give our examples. Define $\bar{A}$ to be the graded algebra generated by the unit 1 and the elements Sq^0, Sq^1, Sq^2, ..., with Sq^i in degree i, subject to the relations

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} \mathrm{Sq}^{a+b-i}\mathrm{Sq}^i \quad \text{if } a<2b. \tag{1}$$

Note that $\mathrm{Sq}^0 \neq 1$, so that $\bar{A}$ is not quite the Steenrod algebra familiar to algebraic topologists. In fact, the degree 0 component of $\bar{A}$ is $\mathbb{Z}_2[\mathrm{Sq}^0]$, the ring of polynomials in Sq^0.

The augmentation $\varepsilon : \bar{A} \to \mathbb{Z}_2$ must send Sq^i to 0 if $i>0$, but $\varepsilon(\mathrm{Sq}^0)$ can be either 0 or 1. We let $\bar{A}_i$ be $\bar{A}$ with augmentation $\varepsilon(\mathrm{Sq}^0)=i$, $i=0$ or 1.

The algebra $\bar{A}_0$ is a homogeneous Koszul algebra with presentation $\alpha(x_i)=\mathrm{Sq}^i$ by [3, Theorem 5.3]. By the above theorem, $H^*\bar{A}_0$ is generated as an algebra by elements λ_i dual to Sq^{i+1}, $i\geq -1$, with relations

$$\lambda_{2i+n+1}\lambda_i = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{2i+j+1}\lambda_{i+n-j} \tag{2}$$

for $n\geq 0$. In particular, $\lambda_{-1}^2 = 0$. This algebra is well known to the inventors of the lambda algebra [1] as the opposite algebra of the algebra obtained from the lambda algebra by adjoining an element λ_{-1}. Let us write Λ^+ for $(H^*\bar{A}_0)^{\mathrm{op}}$. If we write $A_{\mathrm{L}} = \bar{A}_0/(\mathrm{Sq}^0)$, then Priddy showed [3, 7.1] that $\Lambda = (H^*A_{\mathrm{L}})^{\mathrm{op}}$. The quotient homomorphism induces the obvious inclusion $\Lambda \to \Lambda^+$ and 'explains' why it is possible to adjoin λ_{-1} to Λ. This explanation of the origin of λ_{-1} was suggested to me by M. Mahowald.

The algebras $\bar{A}$ and A_L have been identified in [2] and [4] as the Steenrod algebras of cohomology operations for the cohomology of cocommutative Hopf algebras and simplicial restricted Lie algebras respectively. The Steenrod algebra A which operates on the cohomology of topological spaces fits in as follows. A is the quotient augmented algebra (in fact, Hopf algebra) of $\bar{A}_1$ by the monoid ring on Sq^0,

$$\mathbb{Z}_2[\mathrm{Sq}^0] \xrightarrow{\triangleleft} \bar{A}_1 \to A. \tag{3}$$

All three of these are Koszul algebras and if we take the associated homogeneous algebras we obtain the extension

$$\mathbb{Z}_2[\mathrm{Sq}^0] \xrightarrow{\triangleleft} \bar{A}_0 \to A_L, \tag{4}$$

with $\varepsilon(\mathrm{Sq}^0)=0$ in $\mathbb{Z}_2[\mathrm{Sq}^0]$ here. The remarkable fact here is that $E^0\bar{A}_1 \cong \bar{A}_0$ as algebras; filtering has changed only the augmentation. We will prove this in the process of proving the following theorem, which shows that $\bar{A}_0$ and $\bar{A}_1$ are the examples we want:

Theorem. $H^*\bar{A}_1 = E[\lambda_{-1}] \otimes H^*A$.

Proof. To use Priddy's theorem, we must give a presentation $\alpha: T\{x_i\} \to \bar{A}_1$ of *augmented* algebras. The point is that in the tensor algebra, each x_i has augmentation 0, so that Sq^0, having augmentation 1, is not a suitable Koszul generator. Thus, we let $\alpha(x_i)=\mathrm{Sq}^i$ for $i>0$, and let $\alpha(x_0)=\mathrm{Sq}^0-1$. The relations (1) then become

$$\overline{\mathrm{Sq}}^a\overline{\mathrm{Sq}}^b = \binom{b-1}{a}\overline{\mathrm{Sq}}^{a+b} + \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i}\overline{\mathrm{Sq}}^{a+b-i}\overline{\mathrm{Sq}}^i \quad \text{if } a<2b \tag{5}$$

with $\overline{\mathrm{Sq}}^i=\alpha(x_i)$. The homogeneous part of (5) is (1), showing that $E^0\bar{A}_1 \cong \bar{A}_0$. The inhomogeneous terms of (5) induce a differential in $H^*\bar{A}_0=(\Lambda^+)^{\mathrm{op}}$ given by

$$\begin{aligned}\delta\lambda_n &= \sum_{k\geq 0}\binom{n-k-1}{k+1}\lambda_k\lambda_{n-k-1} \\ &= [\lambda_{-1},\lambda_n] = \lambda_{-1}\lambda_n + \lambda_n\lambda_{-1}.\end{aligned} \tag{6}$$

From the latter form it follows that $\delta x=[\lambda_{-1},x]$ for all x. (The fact that $\delta\lambda_n=[\lambda_{-1},\lambda_n]$ was pointed out to me by Bousfield.) Restricting to the lambda algebra we get Priddy's Koszul complex for $H^*A=H(\Lambda^{\mathrm{op}},\delta)$. By the relations (2) and the fact that $\delta(\lambda_{-1})=0$, it follows that $(\Lambda^+)^{\mathrm{op}}=\lambda_{-1}\Lambda^{\mathrm{op}}\oplus\Lambda^{\mathrm{op}}$ as a chain complex. Thus, $H(\Lambda^{+\,\mathrm{op}},\delta)=E[\lambda_{-1}]\otimes H^*A$ as modules, and since $[\lambda_{-1},x]$ is always a boundary, this also holds as algebras.

Since A is a Hopf algebra, H^*A and hence $H^*\bar{A}_1$ are commutative. In contrast, $H^*\bar{A}_0$ is far from commutative.

Note that the Cartan Eilenberg spectral sequence of (3) says that $H^*\bar{A}_1$ is no larger than $E[\lambda_{-1}]\otimes H^*A$ and the spectral sequence therefore collapses. That $E_2=E_\infty$ also follows directly from the obvious splitting homomorphism $A_1\to\mathbb{Z}_2[\mathrm{Sq}^0]$ but the extension question from E_∞ to $H^*\bar{A}_1$ is not as easily settled.

Finally, note that the degree 0 components of $\bar{A}_0$ and $\bar{A}_1$ are the classic examples of the *in*sensitivity of cohomology to the augmentation: $H^*\mathbb{Z}_2[\mathrm{Sq}^0]$ is an exterior algebra on one generator no matter which augmentation we choose. □

References

[1] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector and J.W. Schlesinger, The mod-*p* lower central series and the Adams spectral sequence, Topology 5 (1966) 331–342.

[2] J.P. May, A general algebraic approach to the Steenrod algebra, in: The Steenrod Algebra and its Applications, Lecture Notes in Mathematics 168 (Springer, Berlin, 1970) 153–231.

[3] S. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. 152 (1970) 39–60.

[4] S. Priddy, Primary cohomology operations for simplicial Lie algebras, Illinois J. Math. 14 (1970) 585–612.