THE ADAMS SPECTRAL SEQUENCE FOR THE IMAGE-OF-J SPECTRUM

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ABSTRACT. We show that if we factor the long exact sequence in cohomology of a cofiber sequence of spectra into short exact sequences, then the $d_2$-differential in the Adams spectral sequence of any one term is related in a precise way to Yoneda composition with the 2-extension given by the complementary terms in the long exact sequence. We use this to give a complete analysis of the Adams spectral sequence for the connective image-of-$J$ spectrum, finishing a calculation that was begun by D. Davis in 1975.

1. Introduction

Let

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \]

be a homotopy cofiber sequence of spectra. Let $p$ be a prime, let $H = H\mathbb{F}_p$ be the mod $p$ Eilenberg–MacLane ring spectrum, let $H^*$ denote mod $p$ cohomology, and consider the induced long exact sequence of $A$-modules

\[ \cdots \leftarrow H^* (X) \xrightarrow{f^*} H^* (Y) \xrightarrow{g^*} H^* (Z) \xrightarrow{h^*} \Sigma H^* (X) \leftarrow \cdots \]

where $A = H^*(H)$ denotes the mod $p$ Steenrod algebra. Letting

\[ K = \ker(g^*) \quad I = \im(g^*) \quad C = \cok(g^*) \]

we obtain the following short exact sequences of $A$-modules:

\[ (E_X) \quad 0 \leftarrow \Sigma^{-1} K \xleftarrow{q^*} H^* (X) \xrightarrow{i^*} C \leftarrow 0 \]

\[ (E_Y) \quad 0 \leftarrow C \xleftarrow{\delta_Y^*} H^* (Y) \xrightarrow{i_Y^*} I \leftarrow 0 \]

\[ (E_Z) \quad 0 \leftarrow I \xleftarrow{\delta_Z^*} H^* (Z) \xrightarrow{i_Z^*} K \leftarrow 0 . \]

We can compare the Adams $d_2$-differentials for $X$, $Y$ and $Z$, as in Figure 1. If $g^*$ is injective (so that $K = 0$) or surjective (so that $C = 0$), the composite $i^*d_2g^*$ is necessarily zero. If $g^*$ is trivial (so that $I = 0$), then $i = f^*$ and $q = h^*$, and $i^*d_2q^*$ factors through $fh$, hence is zero. In these cases the Adams $E_2$-terms of $X$, $Y$ and $Z$ form a long exact sequence, and the geometric boundary theorem [Bru78] gives useful information about the relationship between their $d_2$-differentials.

Our first theorem addresses what happens when $g^*$ is neither injective, surjective or zero, in which case $i^*d_2g^*$ can indeed be nontrivial.

Theorem 1.1. Consider the homotopy cofiber sequence (1.1). In the mod $p$ Adams spectral sequence for $X$ the composite homomorphism

\[ i^*d_2g^* : \Ext_A^s (\Sigma^{-1} K, \mathbb{F}_p) \xrightarrow{q^*} \Ext_A^{s,t} (X) \xrightarrow{d_2} \Ext_A^{s+2,t+1} (X) \xrightarrow{i^*} \Ext_A^{s+2,t+1} (C, \mathbb{F}_p) \]

agrees, up to a sign, with the composition

\[ \delta_Y \delta_Z : \Ext_A^s (\Sigma^{-1} K, \mathbb{F}_p) \xrightarrow{\delta_Z^*} \Ext_A^{s+1,t+1} (I, \mathbb{F}_p) \xrightarrow{\delta_Y^*} \Ext_A^{s+2,t+1} (C, \mathbb{F}_p) \]

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of the connecting homomorphisms associated to the extensions \((E_Z)\) and \((E_Y)\).

Furthermore, \(\delta Y\delta Z\) agrees, up to a sign, with the Yoneda product with \(e_Z e_Y\), where \(e_Z \in \Ext^1_A(I, K)\), \(e_Y \in \Ext^1_A(C, I)\) and \(e_Z e_Y \in \Ext^2_A(C, K)\) are the classes associated to the extensions \((E_Z)\), \((E_Y)\) and

\[
0 \longleftarrow C \longleftarrow H^*(Y) \overset{g^*}{\longleftarrow} H^*(Z) \longleftarrow K \longleftarrow 0,
\]

respectively. See [ML63, Thm. III.9.1]. Note that, while the Adams \(d_2\)-differential in \(E_2(X)\) generally depends on the topological structure of the spectrum \(X\), our theorem shows that the composite \(i^* d_2 q^*\) only depends on the algebraic structure given by the long exact sequence (1.2) of \(A\)-modules.

Next we specialize to the (implicitly 2-complete) homotopy (co-)fiber sequence

\[
j \rightarrow ko \overset{\psi}{\rightarrow} \Sigma^4 ksp \rightarrow \Sigma j
\]

defining the connective image-of-\(J\)-spectrum \(j\). Here \(ko\) and \(ksp\) denote connective real and quaternionic \(K\)-theory, respectively, and \(\psi\) denotes a lift of \(\psi^3 - 1: ko \rightarrow ko\) over the 3-connected cover \(b spin \simeq \Sigma^4 ksp\). There is an extension

\[
(E_j) \quad 0 \longleftarrow \Sigma^{-1} K \longleftarrow H^*(j) \overset{j^*}{\longleftarrow} C \longleftarrow 0
\]

with

\[
C \cong A/A(Sq^1, Sq^2, Sq^3) = A//A(2)
\]

\[
K \cong \Sigma^3 A/A(Sq^1, Sq^7, Sq^4 Sq^6 + Sq^6 Sq^4)
\]

and a long exact sequence

\[
\cdots \rightarrow \Ext^{-1,t}_A(C, F_2) \overset{d_\infty}{\rightarrow} \Ext^t_A(C, F_2) \rightarrow \Ext^t_A(C, F_2) \rightarrow \cdots
\]

Using Theorem 1.1 to get a lower bound on the rank of \(d_2: E_2^{s,t}(j) \rightarrow E_2^{s+2,t+1}(j)\), we deduce our second theorem. The \((E_2, d_2)\), \((E_3)\) and \((E_\infty)\) for \(j\) are illustrated in Figures 4, 5 and 6, respectively. The calculation of the Adams spectral sequence for \(j\) begun in 1975 was stymied by the evidently large amount of cancellation which must occur in the spectral sequence. Our theorem shows that this cancellation, which reduces the Krull dimension from 4 to 2, almost all occurs from \(E_2(j)\) to \(E_3(j)\), and is described in a simple fashion.
Theorem 1.2. Let $j$ be the connective image-of-$J$ spectrum at $p = 2$. There is an isomorphism
\[
E_\infty(j) \cong \mathbb{F}_2[w_1, h_1, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0]
\]
\[\oplus \mathbb{F}_2[w_i^2, h_0 h_3] \quad (0 \leq i \leq 3)
\]
\[\oplus (\mathbb{F}_2[h_0, w_i^1]\{1, h_3 w_i^1\} \oplus \mathbb{F}_2[w_i^1]\{h_0 h_3 w_1 \mid 0 \leq i \leq 4\})
\]
with generators in $(s, t)$-bidegrees $|h_0| = (1, 1)$, $|h_1| = (1, 2)$, $|h_2| = (1, 4)$, $|h_3| = (1, 8)$, $|c_0| = (3, 11)$ and $|w_1| = (4, 12)$. The remaining nonzero differentials are
\[d_r(h_0^j w_i^1) = h_0^{j+r+3} h_3 w_1^{k-1}
\]
for $r \geq 3$, $i \geq 0$ and $\text{ord}_2(k) = r - 1$. Hence
\[E_\infty(j) \cong \mathbb{F}_2[h_0]
\]
\[\oplus \mathbb{F}_2[w_1, h_1, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0]
\]
\[\oplus \bigoplus_{r \geq 1} \mathbb{F}_2[w_i^2, h_0 h_3 w_i^{2r-1} \mid 0 \leq i \leq r + 2\}
\]

Here and below we use $\text{ord}_p(k)$ to denote the $p$-adic valuation of $k$.

Remark 1.3. We write $h_3 \in E_2^{1,8}(j)$ for the image of the usual class $h_3 \in E_2^{1,8}(S)$ under the unit map $e : S \to j$. This is also the image $q^*(\Sigma^{-1} h_0 h_2^2 h_3) = q^*(\Sigma^3 v')$ for classes $\Sigma^{-1} h_0 h_2^2 h_3 \in \text{Ext}_A^{1,8}(\Sigma^{-1} K_1; \mathbb{F}_2)$ and $\Sigma^3 v' \in E_2^{1,8}(\Sigma^3 K_1)$. There is a well-defined action by powers of $w_1$ on the latter two Ext-groups. The classes $h_3 w_1^{k-1}$ in $E_2(j)$ are therefore defined to be the images under $q^* i^3_{\mathbb{Z}}$ of the products $\Sigma^3 v' \cdot w_1^{k-1}$. We explain the notation $\mathbb{F}$ in Lemma 3.7, which builds upon a notation $\mathbb{F}$ introduced in Lemma 3.6.

The unit map sends $\mathbb{F}_2[h_0] \subset E_\infty(S)$ isomorphically to $\mathbb{F}_2[h_0] \subset E_\infty(j)$. Direct calculation with $\text{Ext}_A$ shows that each class $x \in \{h_1, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}$ in $E_2(S)$ is mapped by $e$ to the class with the same name in $E_2(j)$. Naturality with respect to the Adams periodicity operator $Px = \langle h_3, h_4, x \rangle$ then shows that $P^{k-1} x$ maps to $x w_1^{k-1}$, for each such $x$ and $k \geq 1$. The classes $P^{k-1} x$ are all infinite cycles by Adams vanishing [Ad46, Thm. 1.1], and Theorem 1.2 shows that the classes $x w_1^{k-1}$ remain nonzero in $E_\infty(j)$. Hence $P^{k-1} x$ and $x w_1^{k-1}$ all survive to nonzero classes in the $E_\infty$-terms, with $e$ mapping the former to the latter.

Remark 1.4. The case of topological degree $8k - 1$ is well known to be significantly more difficult. Davis and Mahowald [DM89, Thm. 1.1] proved that any generator $\rho_{8k-1}$ of the image of the $J$-homomorphism in $\pi_{8k-1}(S)$ is detected in $E_\infty(S)$ by a class of Adams filtration $4k - 3 \equiv \text{ord}_2(k)$, and that this class supports an $h_0$-tower that ends in Adams filtration $4k$. On the other hand, Theorem 1.2 shows that any generator $j_{8k-1}$ of $\pi_{8k-1}(j)$ is detected in $E_\infty(j)$ by $h_3 w_1^{k-1}$ in Adams filtration $4k - 3$, and that this supports an $h_0$-tower of the same height, ending in Adams filtration $4k + \text{ord}_2(k)$. It follows that $e$ maps the image-of-$J$ subgroup in $\pi_{8k-1}(S)$ isomorphically to $\pi_{8k-1}(j)$, but increases the Adams filtration of each class by exactly $\text{ord}_2(k)$.

The hidden $\eta$-extension in $E_\infty(j)$ from $h_3 w_1^{k-1}$ detecting $j_{8k-1}$ to $c_0 w_1^{k-1}$ detecting $\eta_j$-extension in $E_\infty(j)$ from the class detecting $\rho_{8k-1}$ to $P^{k-1} c_0$ detecting $\eta_j \rho_{8k-1}$ (except for $k = 1$), which shifts Adams filtration by the variable amount $2 + \text{ord}_2(k)$.

In Section 4 we make a similar analysis of the Adams spectral sequence for the mod 2 reduction $j/2$ of the image-of-$J$ spectrum, which turns out to collapse at
the $E_3$-term. In Section 5 we summarize the corresponding calculations for the $p$-
primary image-of-$J$ spectrum, where $p$ is any odd prime. See Theorems 4.5 and 5.12
for precise statements, and Figures 7 and 8 for illustrations.

2. Adams differentials in a homotopy cofiber sequence

Our study depends on a standard functorial construction of the Adams spectral
sequence, which we now review. Let

$$\cdots \to S_{s+1} \xrightarrow{\alpha} S_s \to \cdots \to S_1 \xrightarrow{\alpha} S_0 = S$$

be the canonical mod $p$ Adams tower for the sphere spectrum. For each $s \geq 0$ we
have a homotopy cofiber sequence

$$S_{s+1} \xrightarrow{\alpha} S_s \xrightarrow{\beta} H \wedge S_s \xrightarrow{\gamma} \Sigma S_{s+1},$$

where $\beta$ is induced by the ring spectrum unit map $S \to H$. Smashing these diagrams
with $X$ we obtain the canonical Adams tower for $X$, and we set $X_s = S_s \wedge X$. The
mod $p$ Adams spectral sequence for $X$ is the homotopy spectral sequence associated
to the following diagram, where $\gamma$ has degree $-1$.

$$\cdots \to X_{s+1} \xrightarrow{\alpha} X_s \xrightarrow{\beta} \cdots \to X_1 \xrightarrow{\alpha} X$$

When $H^*(X)$ is bounded below and of finite type, the Künneth theorem shows that
the cohomology of the lower part is isomorphic to a free resolution

$$\cdots \to C_s(A, H^*(X)) \xrightarrow{\partial} C_{s-1}(A, H^*(X)) \to \cdots \to C_0(A, H^*(X)) \xrightarrow{\epsilon} H^*(X),$$

where we set

$$C_s(A, M) = A \otimes I(A)^{\otimes s} \otimes M$$

for any $A$-module $M$. As usual, $I(A) = \ker(\epsilon: A \to \mathbb{F}_p)$ denotes the augmentation
ideal. The Hopf algebra $A$ acts diagonally on this $(s + 2)$-fold tensor product,
which is nonetheless free because of the untwisting isomorphism. Furthermore, the
Hurewicz homomorphism induces isomorphisms

$$\pi^{s,t}_{s-1}(H \wedge X_s) \xrightarrow{\cong} \text{Hom}_A(H^*(H \wedge X_s), \Sigma^{s-t}\mathbb{F}_p) \cong C^{s,t}_A(H^*(X), \mathbb{F}_p),$$

where

$$C^{s,t}_A(M, \mathbb{F}_p) = \text{Hom}_A(C_s(A, M), \Sigma^t \mathbb{F}_p).$$

By naturality, it follows that

$$(E_1^{s,t}(X), d_1) \cong (C^{s,t}_A(H^*(X), \mathbb{F}_p), \delta)$$

with $\delta = \text{Hom}(\partial, 1)$, which leads to the familiar formula

$$E_2^{s,t}(X) \cong \text{Ext}_A^{s,t}(H^*(X), \mathbb{F}_p).$$

Remark 2.1. The canonical resolution $C_*(A, M) \to M$ is isomorphic to the normal-
ized bar resolution, and $C^*_A(M, \mathbb{F}_p)$ is isomorphic to the normalized cobar complex,
but we do not need to make these isomorphisms explicit.

Remark 2.2. If $H^*(X)$ is not bounded below and of finite type, we can instead work
with $H_*(X)$ as an $A_*$-comodule, where $A_* = H_*(H)$ is the dual mod $p$ Steenrod
(Hopf) algebra. We set

$$C_*(A_*, M_*) = A_* \otimes J(A_*)^{\otimes s} \otimes M_*$$

and

$$C^{s,t}_A(\mathbb{F}_p, M_*) = \text{Hom}_{A_*}(\Sigma^t \mathbb{F}_p, C_*(A_*, M_*)), $$
where $M_s$ is any $A_s$-comodule and $J(A_s) = \text{cok}(q; \mathbb{F}_p \rightarrow A_s)$. Then
\[
E_1^{s,t}(X) = \pi_{-s}(H \wedge X_s) \cong C_{A_s}^s(\mathbb{F}_p, H_s(X))
\]
and $E_2^{s,t}(X) \cong \text{Ext}_{A_s}^s(\mathbb{F}_p, H_s(X))$, with $\text{Ext}$ formed in $A_s$-comodules. Our results all apply in this generality, but for ease of comparison with the calculations in the next section we prefer to write in terms of cohomology and $A$-modules.

The Adams differential $d_2$: $E_2^{s,t}(X) \rightarrow E_2^{s+2,t+1}(X)$ is given by $d_2([x]) = [\beta(u)]$, where $x \in E_1^{s,t}(X)$ is a $d_1$-cycle, $u \in \pi_{t-s-1}(X_{s+2})$, and $\gamma(x) = \alpha(u)$.

\[
\begin{array}{ccccccc}
\pi_s(X_{s+2}) & \xrightarrow{\alpha} & \pi_s(X_{s+1}) & \xrightarrow{\alpha} & \pi_s(X_s) \\
\downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} \\
E_1^{s,t+2}(X) & \xrightarrow{\alpha'} & E_1^{s+1,t}(X) & \xrightarrow{\alpha'} & E_1^{s,t}(X)
\end{array}
\]

Let $S_{s,r} = \text{cof}(\alpha'': S_{s+r} \rightarrow S_s)$ be the mapping cone, so that $S_{s,1} \simeq H \wedge S_s$ and we have homotopy cofiber sequences
\[
(2.1) \quad S_{s+1} \xrightarrow{\alpha'} S_{s+1,2} \xrightarrow{\beta'} S_{s+1,1} \xrightarrow{\gamma'} \Sigma S_{s+2,1}
\]
\[
(2.2) \quad S_{s+2,1} \xrightarrow{\alpha''} S_{s+3} \xrightarrow{\beta''} S_{s+1,1} \xrightarrow{\gamma''} \Sigma S_{s+1,2}
\]

Letting $X_{s,r} = S_{s,r} \wedge X$ we get similar sequences by smashing these with $X$. Then $d_2([x]) = [v]$ for any $v \in E_1^{s+2,t+1}(X)$ with $\gamma''(x) = \alpha'(v)$ in the following diagram.

\[
E_1^{s+2,t+1}(X) \xrightarrow{\alpha'} \pi_{s}(X_{s+1,2}) \xrightarrow{\gamma''} - E_1^{s,t}(X)
\]

In each case, a choice of $u$ or $v$ exists because $\beta \gamma(x) = d_1(x) = 0$, and the difference between any two choices maps to zero in $E_2^{s+2,t+1}(X)$.

Smashing the continuation of (1.1) and (2.1) together, we obtain the commutative diagram in Figure 2 with horizontal and vertical homotopy cofiber sequences.
up to some signs which we suppress. Applying \( \pi_* \), we obtain a similar diagram with long exact rows and columns.

**Proposition 2.3.** Let \( w \in \pi_n(\Sigma X_{s+1,2}) \) satisfy \( f(w) = 0 \) and \( \beta'(w) = 0 \). Then

\[
\pm f(\alpha')^{-1}(w) = \gamma' g^{-1} \beta' h^{-1}(w)
\]

as subsets of \( \pi_n(\Sigma Y_{s+2,1}) \). The indeterminacy of either expression is the image of \( \pm f \gamma' = \gamma f \).

**Proof.** If \( X, Y \) and (hence) \( Z \) are dualizable, then this follows from the fill-in axiom for the triangulated structure on the stable homotopy category, applied to the diagram below.

The general case follows by a passage to colimits. See also [BG95, Lem. 2.2], and [AM17, Lem. 9.3.2] correcting [Mil81, Lem. 6.7].

**Proof of Theorem 1.1.** The extensions \((E_X), (E_Y)\) and \((E_Z)\) induce short exact sequences of cochain complexes, giving factorizations of the maps of Adams \( E_1 \)-terms induced by \( f \), \( g \) and \( h \), as shown in Figure 3.

![Diagram for the proof of Theorem 1.1](image)
it follows that the cocycles $J$ at the prime $p$ Ext $\delta$ $ko$ the subalgebra generated by Proposition 3.1 choose an $\tilde{\gamma}$ $w$ associated to $(E f)$ a possible sign.

Next choose a $\tilde{\gamma}$ $(A s)$ lifts uniquely over $q Z$ to a cocycle $z \in C^{s+1,t+1}(I,F_p)$, as in the definition of the connecting homomorphism

$$\delta_Z: \text{Ext}_A^s(\Sigma^{-1}K,F_p) \to \text{Ext}_A^{s+1,t+1}(I,F_p)$$

associated to the short exact sequence of cochain complexes induced by $(E_Z)$. Hence $[z] = \delta_Z[x]$. Next choose a $\tilde{z} \in E^{s+1,t+1}(Y)$ that maps by $i^*_Y$ to $z$. Then $q Z \tilde{z} = d_1(\tilde{z})$, so $\tilde{z} \in g^{-1}\beta h^{-1}(w)$. The image $\gamma' \tilde{z} = d_1(\tilde{z}) = y$ in $E^{s+2,t+1}(Y)$ then lies in the coset $\gamma' g^{-1}\beta h^{-1}(w)$. Moreover, since $z$ is a cocycle, $\tilde{y}$ lifts uniquely over $q Z$ to a cocycle $y \in C^{s+2,t+1}(C,F_p)$, as in the definition of the connecting homomorphism

$$\delta_Y: \text{Ext}_A^{s+1,t+1}(I,F_p) \to \text{Ext}_A^{s+2,t+1}(C,F_p)$$

associated to $(E_Y)$. Hence $[y] = \delta_Y[z]$.

We have now shown that $i^* d_2 q^* [x]$ and $\delta_Y \delta_Z[x]$ in $\text{Ext}_A^{s+2,t+1}(C,F_p)$ are represented by cocycles $i^* v$ and $y$ in $C^{s+2,t+1}(C,F_p)$, respectively, which have the same image in $E^{s+2,t+1}(Y)$, up to a sign and modulo the image of $f \gamma'$. In view of the commutative diagram

$$\begin{array}{ccc}
E_1^{s+2,t+1}(X) & \xrightarrow{\gamma'} & E_1^{s+2,t+1}(X) \\
\downarrow{i^*} & & \downarrow{i^*} \\
C^{s+2,t+1}(C,F_p) & \xrightarrow{\delta} & C^{s+2,t+1}(C,F_p) \\
\downarrow{q^*_Y} & & \downarrow{q^*_Y} \\
E_1^{s+1,t+1}(Y) & \xrightarrow{\gamma'} & E_1^{s+2,t+1}(Y)
\end{array}$$

it follows that the cocycles $i^* v$ and $y$ agree in $C^{s+2,t+1}(C,F_p)$, up to a sign and modulo the image of $\delta$, i.e., up to a coboundary. In other words, $[i^* v] = \pm [y]$ in $\text{Ext}_A^{s+2,t+1}(C,F_p)$. \hfill \Box

3. The image-of-J spectrum

We now specialize to the case when the homotopy cofiber sequence (1.1) is the sequence (1.3) defining the connective image-of-J spectrum, implicitly completed at the prime $p = 2$. The mod 2 cohomology and Adams spectral sequence for the ring spectrum $ko$ and its module spectrum $ksp$ are well-known. Let $A(n) \subset A$ be the subalgebra generated by $S q^1, \ldots, S q^2$.

**Proposition 3.1** ([Sto63], [BR21, §2.6]).

(1) $H^*(ko) = A/A(S q^1, S q^2) = A/\langle 1 \rangle$ and

$$E_2(ko) = \text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, v, w_1]/(h_0 h_1, h_0^2, h_1 v, v^2 + h_0^2 w_1)$$

with algebra generators in $(s,t)$-bidegrees $|h_0| = (1,1)$, $|h_1| = (1,2)$, $|v| = (3,7)$ and $|w_1| = (4,12)$. 

(2) \( \pi_4(ko) = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 - 4B) \) with \( |\eta| = 1, |A| = 4 \) and \( |B| = 8 \).

(3) \( H^*(ksp) = A/A(S^3A, Sq^2Sq^3) \) and
\[
E_2(ksp) = E_2(ko\{1, v\}'/(h_1 \cdot 1, v \cdot 1 + h_0 \cdot v, v \cdot v' + w_1 \cdot 1)
\]
with module generators in \((s, t)\)-bidegrees \(|1| = (0, 0)\) and \(|v'| = (1, 5)\).

The action of the Adams operation \( \psi^3 \) on the homotopy of \( ko \), and the resulting homotopy groups of the image-of-\( J \) spectrum, are also well-known.

**Proposition 3.2** ([Ada62], [Dav75, Prop. 2], [BR21, §11.3]).

1. \( \psi^3(\eta) = \eta, \psi^3(A) = 3^2A \) and \( \psi^3(B) = 3^4B \).
2. \( \pi_0(j) = \mathbb{Z}, \pi_1(j) = \mathbb{Z}/2[\eta] \) and, when \( n \geq 2 \),
\[
\pi_n(j) = \begin{cases} 
\mathbb{Z}/16k & \text{for } n = 8k - 1, \\
\mathbb{Z}/2 & \text{for } n \equiv 0 \mod 8, \\
(\mathbb{Z}/2)^2 & \text{for } n \equiv 1 \mod 8, \\
\mathbb{Z}/2 & \text{for } n \equiv 2 \mod 8, \\
\mathbb{Z}/8 & \text{for } n \equiv 3 \mod 8, \\
0 & \text{otherwise.}
\end{cases}
\]

The \( A \)-module homomorphism \( \psi^* \) is induced up from the subalgebra \( A(2) \), so the calculation of its kernel, image and cokernel is a finite algebraic problem.

**Proposition 3.3** ([Dav75, Lem. 3], [AR05, Lem. 7.6(c)]).

1. \( \psi^*: H^*(\Sigma^4ksp) \to H^*(ko) \) is determined by \( \psi^*(\Sigma^41) = Sq^4 \).
2. \( K = \ker(\psi^*) = \Sigma^4A/A(Sq^4, Sq^7, Sq^4Sq^6 + Sq^6Sq^4) \).
3. \( I = \im(\psi^*) = \Sigma^4A/A(Sq^4, Sq^6) \).
4. \( C = \cok(\psi^*) = A/A(Sq^4, Sq^2, Sq^4) = A/A(2) \).

**Proof.** Only case (3) may be new. It amounts to the claim that the \( A(2) \)-submodule of \( A(2)/A(1) \) generated by \( Sq^4 \) has annihilator ideal \( A(2)/(Sq^4, Sq^6) \), which can be checked by direct calculation. This is also implicit in [DM82, Thm. 5.9]. \( \square \)

We will see that \( \text{Ext}_A(K, \mathbb{F}_2) \), \( \text{Ext}_A(I, \mathbb{F}_2) \) and \( \text{Ext}_A(C, \mathbb{F}_2) \) are closely related, where the latter is explicitly known.

**Proposition 3.4** ([Sl67], [BR21, §3.5]).

\[
\text{Ext}_A(C, \mathbb{F}_2) = \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, w_1, \alpha, \beta, d_0, e_0, g, \gamma, \delta, w_2]/(\sim)
\]
is a free \( \mathbb{F}_2[w_1, w_2] \)-module, where \( (\sim) = (h_0h_1, \ldots, \delta^2) \) denotes an ideal generated by 54 explicit relations. The generators are graded as follows.

| \( m \) | \( h_0 \) | \( h_1 \) | \( h_2 \) | \( c_0 \) | \( w_1 \) | \( \alpha \) | \( \beta \) | \( d_0 \) | \( e_0 \) | \( g \) | \( \gamma \) | \( \delta \) | \( w_2 \) |
|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 0 | 1 1 | 1 1 | 3 4 | 4 3 | 3 4 | 4 4 | 4 5 | 5 7 | 8 |
| t 0 | 1 2 | 4 11 | 19 12 | 15 18 | 18 21 | 24 30 | 59 |
| t - s | 0 1 | 3 8 | 8 12 | 15 14 | 17 20 | 25 32 | 48 |

**Remark 3.5.** The \( A \)-modules \( H^*(ko) \) and \( H^*(ksp) \) are induced up from \( A(1) \), while the modules \( K \), \( I \) and \( C \), as well as the extensions \( (E_{ko}) \) and \( (E_{ksp}) \), below, are induced up from \( A(2) \). Hence the long exact sequences in Lemmas 3.6 and 3.7 consist of \( \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \)-modules and \( \psi \)-homomorphisms. In particular, \( w_1 \) acts linearly on these sequences. The module \( H^*(j) \) and the extension \( (E_j) \) are not induced up from \( A(2) \), but from \( A(3) \), so the long exact sequence in the proof of Theorem 1.2 (below) is \( \text{Ext}_{A(3)}(\mathbb{F}_2, \mathbb{F}_2) \)-linear. It then follows from [Ada66, Lem. 4.4] that \( w_1^2 \) acts linearly on that sequence. Alternatively, we may use that
Lemma 3.6. The $A$-module extension
\[(E_{ka})\quad 0 \leftarrow C \overset{\delta_Y}{\longrightarrow} H^*(ka) \overset{\delta_{\Sigma}}{\longrightarrow} I \leftarrow 0\]
induces a long exact sequence
\[\cdots \to E_2^{-s,t}(ko) \overset{cok}{\longrightarrow} \text{Ext}_A^{-s,t}(I,F_2) \overset{\delta_Y}{\longrightarrow} \text{Ext}_A^{-s,t}(C,F_2) \overset{q_{\Sigma}}{\longrightarrow} E_2^{s,t}(ko) \to \cdots\]

with
\[F_2[h_0,w_1]{v} \cong \ker(\delta_Y) \rightarrow \text{Ext}_A^{*}(I,F_2)\]
and
\[\text{Ext}_A^{*}(C,F_2) \rightarrow \text{cok}(\delta_Y) \cong F_2[h_0,w_1] \oplus F_2[w_1]{h_1,h_2} \].

For each nonzero $x \in \text{Ext}_A^{*-1,t}(C,F_2)$ that maps trivially to $\text{cok}(\delta_Y)$ there is a unique lift $\tau \in \text{Ext}_A^{*-1,t}(I,F_2)$ satisfying $\delta_Y(\tau) = x$, and $\text{Ext}_A(I,F_2)$ consists of these $\tau$, together with the free $F_2[h_0,w_1]$-module on $i_\Sigma^*(v) = h_0h_1h_2$.

Proof. The homomorphism $q_{\Sigma}^*: \text{Ext}_A(C,F_2) \rightarrow \text{Ext}_A(ko)$ equals the restriction homomorphism $\text{Ext}_A(0)(F_2,F_2) \rightarrow \text{Ext}_A(1)(F_2,F_2)$, which sends $h_0, h_1$ and $w_1$ to the elements with the same names, and which sends the remaining algebra generators $h_2, \ldots, w_2$ to zero, because the corresponding target bidegrees are trivial. Hence this algebra homomorphism has image $F_2[h_0,w_1] \oplus F_2[w_1]{h_1,h_2}$ and cokernel $F_2[h_0,w_1]{v}$. The uniqueness of the nonzero lifts $\tau$ follows from the fact that $\text{Ext}_A(C,F_2) = 0$ for those bidegrees in which $\ker(\delta_Y)$ is nonzero, that is, when $t-s = 8k + 3$ and $s \geq 4k + 4$. This is an immediate consequence of the presentation from [BR21, Prop. 3.45] of $\text{Ext}_A(ko,F_2)$ as a direct sum of cyclic $R_0$-modules, where $R_0 = F_2[g,w_1]$. See also [BR21, Fig. 3.12, 3.13]. The relation $i_\Sigma^*(v) = h_0h_1h_2$ holds, as can be verified by direct calculation in $\text{Ext}_A(I,F_2)$ or deduced as part of the next proof.

Lemma 3.7. The $A$-module extension
\[(E_{ksp}^{\Sigma})\quad 0 \leftarrow I \overset{q_{\Sigma}}{\longrightarrow} H^*(\Sigma ko) \overset{\delta_{\Sigma}}{\longrightarrow} K \leftarrow 0\]
induces a long exact sequence
\[\cdots \to E_2^{-s,t}(\Sigma ko) \overset{i_{\Sigma}}{\longrightarrow} \text{Ext}_A^{-s,t}(K,F_2) \overset{\delta_{\Sigma}}{\longrightarrow} \text{Ext}_A^{-s,t}(I,F_2) \overset{q_{\Sigma}}{\longrightarrow} E_2^{s,t}(\Sigma ko) \to \cdots\]

with
\[\Sigma ko [h_0,w_1]{v} \cong \ker(\delta_{\Sigma}) \rightarrow \text{Ext}_A^{*}(K,F_2)\]
and
\[\text{Ext}_A^{*}(I,F_2) \rightarrow \text{cok}(\delta_{\Sigma}) \cong \Sigma ko [h_0,w_1] \oplus F_2[w_1]{h_1v',h_2v'} \].

The submodule $F_2[h_0,w_1]{h_1v',h_2v'}$ of $\text{Ext}_A(I,F_2)$ maps isomorphically to $\text{cok}(\delta_{\Sigma})$. For each nonzero $\pi \in \text{Ext}_A^{*-1,t}(I,F_2)$ that maps trivially to $\text{cok}(\delta_{\Sigma})$ there is a unique lift $\tau \in \text{Ext}_A^{*-1,t}(K,F_2)$ satisfying $\delta_{\Sigma}(\tau) = \pi$, and $\text{Ext}_A(K,F_2)$ consists of these $\tau$, together with the free $F_2[h_0,w_1]$-module on $i_{\Sigma}(\Sigma ko v') = h_0h_1h_2$.

Proof. We must prove that the image of
\[q_{\Sigma}: \text{Ext}_A(I,F_2) \rightarrow E_2(\Sigma ko)\]
is exactly $\Sigma ko [h_0,w_1] \oplus F_2[w_1]{h_1v',h_2v'}$. From the $A$-module presentation of $K$ it follows that $\text{Ext}_A^{*-1,t}(K,F_2) = 0$ for $t-s < 8$ and for $(s,t) = (1,11)$. By exactness, $h_2$ and $\Sigma ko$ in $\text{Ext}_A(I,F_2)$ must map nontrivially to $\Sigma ko$ and $\Sigma ko h_1 v'$ in $E_2(\Sigma ko)$. By $h_0$-linearity this implies that $h_0' \cdot h_2$ maps to $\Sigma ko h_0$ for all $i \geq 0$. 

(E) is dual to a square-zero extension of $A_*$-comodule algebras, which implies that $\delta_X$ is a derivation.
By $h_1$-linearity it also implies that $h_1 \overline{v}$ maps to $\Sigma^4 h_2' v'$. These results propagate $w_1$-linearly, and show that the image is at least as large as claimed.

To show that it is no larger, we use the presentation from [BR21, Prop. 3.45] of $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ as a direct sum of cyclic $R_0$-modules, where $R_0 = F_2[g, w_1, w_2]$. By inspection, each $R_0[h_0]$-module generator of $\text{Ext}_A(I, \mathbb{F}_2)$, other than $h_1 \overline{v}$ and $h_1 \overline{w}$, maps to a trivial bidegree of $E_2(\Sigma^k ksp)$. Hence all of these generators map to zero, and the image from $\text{Ext}_A(I, \mathbb{F}_2)$ does not meet $\Sigma^4 F_2[h_0, w_1]\{v'\}$.

The uniqueness of the nonzero lifts $\overline{v}$ follows, as in the proof of Lemma 3.6, from the fact that $\text{Ext}_{A(1)}^*(C, F_2) = 0$ when $t - s = 8k + 6$ and $s \geq 4k + 3$. From the presentation of $K$, with a generator in degree 8 that is annihilated by $Sq^1$, we see that $i_{2\Sigma}(\Sigma^4 v') = h_0 \cdot \overline{h_2}$ is nonzero in $\text{Ext}_{A(1)}^9(K, F_2)$.

**Lemma 3.8.**

1. The kernel of $\delta_Y \delta_Z$: $\text{Ext}_{A(1)}^*(K, \mathbb{F}_2) \to \text{Ext}_{A(1)}^{*+2}(C, \mathbb{F}_2)$ is 
   $$ \ker(\delta_Y \delta_Z) = F_2[h_0, w_1]\{h_0 \overline{h_2'}\} \cong \Sigma^4 F_2[h_0, w_1]\{v'\}. $$

2. The cokernel of $\delta_Y \delta_Z$: $\text{Ext}_{A(1)}^{*+2}(K, \mathbb{F}_2) \to \text{Ext}_{A(1)}^*(C, \mathbb{F}_2)$ is 
   $$ \text{cok}(\delta_Y \delta_Z) = F_2[h_0, w_1] \oplus F_2[w_1]\{h_1, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}. $$

**Proof.** In the exact kernel–cokernel sequence

$$ 0 \to \ker(\delta_Z) \to \ker(\delta_Y \delta_Z) \xrightarrow{\delta_Z} \ker(\delta_Y) \xrightarrow{\delta_Y} \text{cok}(\delta_Z) \to \text{cok}(\delta_Y) \to 0 $$

of $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$-modules, the connecting homomorphism $\delta$ can be identified with the composition

$$ F_2[h_0, w_1]\{v\} \cong \ker(\delta_Y) \to \text{Ext}_{A(1)}^*(I, \mathbb{F}_2) \xrightarrow{\delta_Y} \text{cok}(\delta_Z) \cong \Sigma^4(\mathbb{F}_2[w_1] \oplus F_2[w_1]\{h_1 v', h_2^2 v'\}), $$

which maps $v$ via $h_0 \cdot \overline{h_2}$ to $\Sigma^4 h_3$. This is injective, and therefore $\ker(\delta_Z) = \ker(\delta_Y \delta_Z)$.

Since $\delta_Y(\overline{h_2}) = h_2$ and $\delta_Y(\overline{w}) = c_0$, the image $\delta_Y(\text{cok}(\delta_Z))$ is the submodule $F_2[w_1]\{h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}$ in $\text{cok}(\delta_Y \delta_Z)$.

Combining these lemmas, $\text{Ext}_{A(1)}^*(K, \mathbb{F}_2)$ is obtained in three steps from the algebra $\text{Ext}_{A(1)}^*(C, \mathbb{F}_2)$ by

- taking the kernel of the map onto $F_2[h_0, w_1] \oplus F_2[w_1]\{h_1, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}$,
- shifting by $(s, t) = (-2, 0)$ to lift over $\delta_Y \delta_Z$, denoting the lift of $x$ by $\overline{x}$, and
- extending by $\Sigma^4 F_2[h_0, w_1]\{i_{2\Sigma}(v')\}$ above $\overline{h_2}$, that is, $i_{2\Sigma}(\Sigma^4 v') = h_0 \overline{h_2}$.

We do not introduce notation for the identification

$$ \text{Ext}_{A(1)}^{*+1}(K, \mathbb{F}_2) = \text{Ext}_{A(1)}^*(\Sigma^{-1} K, \mathbb{F}_2), $$

but note that

$$ \ker(\delta_Y \delta_Z) = \Sigma^{-1} F_2[h_0, w_1]\{h_0 \overline{h_2}\} \cong \Sigma^3 F_2[h_0, w_1]\{v'\} $$

when bigraded as in the right hand expression, and that

$$ \overline{h_2} \in \text{Ext}_{A(1)}^0(K, \mathbb{F}_2) \cong \text{Ext}_{A(1)}^0(\Sigma^{-1} K, \mathbb{F}_2). $$
**Proposition 3.9** ([Dav75, Thm. 1(i)], [AR05, Lem. 7.10(c)]). The $A$-module extension

$$(E_j) \quad 0 \to C \xrightarrow{i} H^*(j) \xrightarrow{q} \Sigma^{-1}K \to 0$$

is given by

$$H^*(j) = \frac{A(g_0, g_7)}{A(Sq^1g_0, Sq^2g_0, Sq^4g_0, Sq^5g_0 + Sq^7g_7, Sq^8g_7, (Sq^4Sq^6 + Sq^6Sq^4)g_7)}$$

with $i(1) = g_0$ and $q(g_7) = \Sigma^71$. In particular, it is induced up from an extension over $A(3)$.

**Proof.** We reproduce Davis’ argument: The extension of $\Sigma^{-1}K$ is determined by the values of $Sq^1g_7 \in \{0, Sq^8g_0\}$, $Sq^7g_7 \in \{0, Sq^{14}g_0\}$ and $(Sq^4Sq^6 + Sq^6Sq^4)g_7 \in \{0\}$. Here $Sq^1Sq^7 = 0$ and $Sq^1Sq^4g_0 = Sq^{15}g_0 \neq 0$, so $Sq^7g_7 = 0$. This leaves at most two possible extensions: the split one with $Sq^1g_7 = 0$ and a nonsplit one with $Sq^7g_7 = Sq^8g_0$. If $(E_j)$ were split, the resulting Adams $E_2$-term would imply that $\pi_7(j)$ had order divisible by $32$. This contradicts Proposition 3.2, and therefore $Sq^7g_7 = Sq^8g_0$, as asserted. \hfill $\square$

**Proof of Theorem 1.2.** The extension $(E_j)$ induces a long exact sequence

$$\cdots \to \Ext_{A}^{s,t}(C, F_2) \xrightarrow{\delta_X} \Ext_{A}^{s,t}(\Sigma^{-1}K, F_2) \xrightarrow{\pi} E_2^{s,t}(j) \xrightarrow{i^*} \Ext_{A}^{s,t}(C, F_2) \to \cdots$$

of modules over $\Ext_{A(3)}(F_2, F_2)$. We claim that

$$\ker(\pi) \subset \Ext_{A(3)}(F_2, F_2) = \F_2[\Sigma^{-1}].$$

From $q^*\delta_X = 0$ and Theorem 1.1 we see that $\ker(\delta_X) \subset \ker(\delta_Y)$, which is $h_0$-torsion free by Lemma 3.8. Hence $\delta_X$ vanishes on the $h_0$-power torsion in $\Ext_{A(3)}^{s,t}(C, F_2)$, and we only need to determine its values on monomials in $w_1$, $\alpha$ and $w_2$, since the remaining algebra generators in Proposition 3.4 are $h_0$-power torsion classes. Furthermore, by $w_2^4$-linearity it suffices to determine $\delta_X(w_1^a\alpha^b\omega^c)$ for $a \in \{0, 1\}$, $b \geq 0$ and $c \geq 0$.

The bigdeg containing $\delta_X(w_1)$ equals $\F_2[\Sigma^{-1}h_0\omega^2] = \F_2[i_2^*(\Sigma^3\omega^c)]$. If $\delta_X(w_1)$ were zero, then $q^*(\Sigma^{-1}h_0^i\omega^2)$ for $0 \leq i \leq 4$ would survive to $E_\infty(j)$ as nonzero classes, so that $\pi_7(j)$ would have order divisible by $32$. This contradicts Proposition 3.2, and proves that $\delta_X(w_1) = \Sigma^{-1}h_0^i\omega^2$. (Alternatively, we can calculate enough of $\Ext_{A(3)}(H^*(j), F_2)$, to see that the $h_0$-tower starting in $\Ext_{A(3)}^{0,0}(H^*(j), F_2)$ has height exactly $5$.) On the other hand, $\delta_X$ maps each of the remaining monomials $w_1^a\alpha^b\omega^c$ with $a \in \{0, 1\}$ to bigdegues where $\ker(\delta_Y)$ is trivial. Hence the image of $\delta_X$ is as claimed.

This shows that the submodule

$$\ker(i^*) \to \Ext_{A(3)}^{s,t}(C, F_2)$$

is given by removing $\F_2[h_0, w_1^2][w_1]$. It also shows that the quotient module

$$\Ext_{A(3)}^{s,t}(\Sigma^{-1}K, F_2) \to \coin(q^*)$$

is given by truncating the $h_0$-tower containing $\Sigma^{-1}h_0\omega^2 = i_2^*(\Sigma^3\omega^c)$ by setting $h_0^i \cdot \Sigma^{-1}\omega^2$ equal to zero, repeated $w_2^4$-periodically. We get a short exact sequence

$$0 \to \coin(q^*) \to E_2(j) \to \ker(i^*).$$

where the projection $\coin(q^*) \to \coin(\delta_Y)$ has kernel equal to the quotient

$$\Sigma^{-1}(\F_2[w_1^2][h_0, h_0^{-1}, h_0^2, h_0^3, h_0^4] \oplus \F_2[h_0, w_1^2][w_1 \cdot h_0^{-1}]).$$
of \( \ker(\delta_Y \delta_Z) \), and the inclusion \( \im(\delta_Y \delta_Z) \hookrightarrow \im(i^*) \) has cokernel
\[
\mathbb{F}_2[h_0, w_1^2] \oplus \mathbb{F}_2[w_1]\{h_1, h_1^2, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}
\]
contained in \( \cok(\delta_Y \delta_Z) \). Additively the structure of \( E_2(j) \) is the same as that given in [Dav75, Thm. 1(ii)]. From the relation \( Sq^8 g_0 + Sq^4 g_7 \) in Proposition 3.9, it follows that \( h_0 \cdot q^* (\Sigma^{-1} \overline{h_2}) = h_3 \in E_2^{-1,8}(j) \), and we adopt this notation now.

By Theorem 1.1, we know that \( d_2^t: E_2^{s,t} j \to E_2^{s+2,t+1}(j) \) has rank no less than the dimension of \( \im(\delta_Y \delta_Z) \) in bidegree \( (s, t) \), and \( d_2^{-2,t-1}: E_2^{-2,t-1}(j) \to E_2^{-t}(j) \) has rank no less than the dimension of \( \im(\delta_Y \delta_Z) \) in the same bidegree. Hence the dimension of \( E_3^{s,t}(j) \) in each bidegree is bounded above by the corresponding dimension for
\[
\mathbb{F}_2[h_0, w_1^2, h_3 w_1] \oplus \mathbb{F}_2[w_1]\{h_1, h_1^2, h_2, h_0 h_2, h_0^2 h_2, c_0, h_1 c_0\}
\]
\[
\oplus \mathbb{F}_2[w_1^2]\{h_3, h_0 h_3, h_0^2 h_3, h_0^3 h_3\}.
\]
See Figure 5 for a picture of this upper bound.

Since we know that \( \pi_{sk-1}(j) \cong \mathbb{Z}/16k \) for \( k \geq 1 \) (modulo odd torsion) there must also be \( d_2 \)-differentials
\[
d_2(w_1^k) = h_0^3 h_3 w_1^{k-1}
\]
for positive \( k \equiv 2 \mod 4 \), while \( d_2(w_1^k) = 0 \) for \( k \equiv 0 \mod 4 \). Hence the upper bound for the dimension of \( E_3(j) \) is not exactly attained. In particular, the rank of \( d_2 \) is one larger than the rank of \( \delta_Y \delta_Z \) in some bidegrees for \( t - s \equiv 16 \mod 32 \).

The remaining differential pattern also follows from the known order of these homotopy groups, since the length of the differential on \( w_1^k \) determines the order of \( \pi_{sk-1}(j) \), and vice versa. \[\square\]

Remark 3.10. In view of the extensions \( h_0 \cdot q^* (\Sigma^{-1} \overline{h_2}) = h_3 \) and \( h_0^3 \cdot d_0 = h_3^2 w_1 \), it follows that \( E_2(j) \) contains an \( h_0 \)-tower of height 5 generated by \( q^* (\Sigma^{-1} \overline{h_2}) \) and \( h_0 \)-towers of height 7 generated by \( q^*(\Sigma^{-1} d_0 w_1^3) \) for odd \( a \geq 1 \), together with infinite \( h_0 \)-towers on \( q^*(\Sigma^{-1} d_0 w_1^3) \) for even \( a \geq 0 \). These extra classes near the bottom of the \( h_0 \)-towers support \( d_2 \)-differentials, hence are no longer present at the \( E_3 \)-term.

Remark 3.11. The map \( e: E_2(S) \to E_2(j) \) of mod 2 Adams spectral sequences detects some \( d_2 \)-differentials in the domain, including \( d_2(x_0) = h_0^3 e_0 \) and \( d_2(h_0) = h_0^3 P d_0 \). These differentials easily propagate to determine those on \( e_0, i, j, k, t, m \) and \( y \), in the usual notation [Tan70], [BR21, Ch. 11].

4. The mod 2 image-op-J spectrum

An analysis similar to, but much easier than, the preceding leads to the conclusion that the Adams spectral sequence for the mod 2 reduction \( j/2 \) of the connective image-op-J spectrum has \( E_4 \)-term as shown in Figure 7. It is then immediate that \( E_3(j/2) = E_{\infty}(j/2) \). A summary of the argument follows.

The mod 2 Moore spectrum \( S/2 = S \cup_2 e^1 \) has cohomology \( H^*(S/2) = E[Sq^1] \), where \( E[-] \) denotes the exterior algebra. Smashing (1.3) with this Moore spectrum, we obtain a homotopy (co-)fiber sequence
\[
\begin{align*}
(4.1) & \quad j/2 \longrightarrow ko/2 \overset{\psi/2}{\longrightarrow} \Sigma^4 ko/2 \longrightarrow \Sigma j/2,
\end{align*}
\]
which we view as a case of the homotopy cofiber sequence (1.1).

The mod 2 cohomology \( A \)-modules and associated Ext groups for the \( ko \)-module spectra \( ko/2 \) and \( ksp/2 \) are easily calculated from Proposition 3.1. As is well known, \( \Ext_A(H^*(ko/2), \mathbb{F}_2) \) consists of a “lightning flash” repeated \( \nu_1 \)-periodically, while \( \Ext_A(H^*(ksp/2), \mathbb{F}_2) \cong \Ext_A^{s+1,t+1}(H^*(ko/2), \mathbb{F}_2) \). We make these explicit
in Proposition 4.1. Similarly, the kernel $K_2 = K \otimes E[Sq^1]$, image $I_2 = I \otimes E[Sq^1]$ and cokernel $C_2 = C \otimes E[Sq^1]$ of $(\psi/2)^*$ can readily be presented as $A$-modules. However, we only need their connectivities, which are immediate from Proposition 3.1.

As in the integral case, $\text{Ext}_A(K_2, \mathbb{F}_2)$, $\text{Ext}_A(I_2, \mathbb{F}_2)$ and $\text{Ext}_A(C_2, \mathbb{F}_2)$ are closely related, where the latter is explicitly known.

**Proposition 4.1** ([BR21, Prop. 4.2]). As an $\mathbb{F}_2[h_0, h_1, w_1]$-module

$$\text{Ext}_A(C_2, \mathbb{F}_2) = \text{Ext}_{A[2]}(E[Sq^1], \mathbb{F}_2)$$

contains

1. a summand isomorphic to $\text{Ext}_A(H^*(ko/2), \mathbb{F}_2)$ that is free over $\mathbb{F}_2[w_1]$ on classes forming a “lightning flash”

<table>
<thead>
<tr>
<th></th>
<th>$i(1)$</th>
<th>$i(h_1)$</th>
<th>$i(h_1^2)$</th>
<th>$\tilde{h}_1$</th>
<th>$h_1\tilde{h}_1$</th>
<th>$h_1^2\tilde{h}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$t$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>$t - s$</td>
<td>0</td>
<td>1</td>
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</tbody>
</table>

and

2. a summand isomorphic to $\Sigma^{1,0} \text{Ext}_A(H^*(\Sigma^4 ksp/2), \mathbb{F}_2)$ that is free over $\mathbb{F}_2[w_1]$ on classes forming a “shifted lightning flash”

<table>
<thead>
<tr>
<th></th>
<th>$i(h_2)$</th>
<th>$\tilde{h}_2$</th>
<th>$h_1\tilde{h}_2$</th>
<th>$h_1^2\tilde{h}_2$</th>
<th>$\tilde{c}_0$</th>
<th>$h_1\tilde{c}_0$</th>
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<tr>
<td>$s$</td>
<td>1</td>
<td>2</td>
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<td>4</td>
<td>4</td>
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</tr>
<tr>
<td>$t$</td>
<td>4</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>$t - s$</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

together with other direct summands.

**Remark 4.2.** In the shifted lightning flash, we have $w_1 \cdot i(h_4) = h_1^2 \tilde{c}_0$.

**Remark 4.3.** The notation in Proposition 4.1 is that of [BR21, §4.1]: $i(x)$ denotes the image of a class $x$ from the bottom cell of $S \cup_2 e^1$, while $\tilde{x}$ projects to a class $x$ on the top cell.

**Lemma 4.4.** The composite

$$\delta_Y : \text{Ext}^{*,*}_A(K_2, \mathbb{F}_2) \longrightarrow \text{Ext}^{*,*+2}_A(C_2, \mathbb{F}_2)$$

is a monomorphism, with cokernel

$$\mathbb{F}_2[w_1] \{i(1), i(h_1), i(h_1^2), \tilde{h}_1, h_1\tilde{h}_1, h_1^2\tilde{h}_1, i(h_2), \tilde{h}_2, h_1\tilde{h}_2, h_1^2\tilde{h}_2, \tilde{c}_0, h_1\tilde{c}_0 \}.$$

**Proof.** By Proposition 3.3 the module $I_2$ is 3-connected and the module $K_2$ is 7-connected. Therefore, $q_Y^* : \text{Ext}^{*,*+2}_A(C_2, \mathbb{F}_2) \to E_2^{s,t}(ko/2)$ is an isomorphism for $t - s \leq 2$, so that $q_Y^*(i(1))$ is nonzero. Then, by $h_0\gamma$, $h_1\gamma$- and $w_1\gamma$-linearity, $q_Y^*$ is an epimorphism, and hence $\delta_Y$ is a monomorphism with cokernel the module in case (1) of Proposition 4.1. Similarly, $q_Z^* : \text{Ext}^{*,*}_A(I_2, \mathbb{F}_2) \to E_2^{s,t}(\Sigma^4 ksp/2)$ is an isomorphism for $t - s \leq 6$. Again by $h_0\gamma$, $h_1\gamma$- and $w_1\gamma$-linearity, $q_Z^*$ is an epimorphism, and hence $\delta_Z$ is a monomorphism with cokernel mapping isomorphically by $\delta_Y$ to the module in case (2) of Proposition 4.1.

**Theorem 4.5.** Let $j/2 = j \wedge S/2$ be the mod 2 connective image-of-J spectrum. There is an isomorphism

$$E_3(j/2) = E_\infty(j/2) \cong \mathbb{F}_2[w_1] \{i(1), i(h_1), i(h_1^2), \tilde{h}_1, h_1\tilde{h}_1, h_1^2\tilde{h}_1, i(h_2), \tilde{h}_2, h_1\tilde{h}_2, h_1^2\tilde{h}_2, \tilde{c}_0, h_1\tilde{c}_0 \}.$$

The bidegrees of these generators are as in Proposition 4.1.
Proof. By Theorem 1.1, \( i^*d_\ast q^* = \delta_Y \delta_Z \) and by Lemma 4.4 this is a monomorphism. Therefore, \( q^* \) is a monomorphism and the sequence

\[
0 \to \text{Ext}^r_{\mathbb{Z}}(\Sigma^{-1} K_2, \mathbb{F}_2) \xrightarrow{q^*} \text{Ext}^r_{\mathbb{Z}}(\Sigma^{-1} K_2, \mathbb{F}_2) \to 0
\]

is short exact. It follows from Theorem 1.1 that \( \text{cok}(\delta_Y \delta_Z) \) is an upper bound for \( E_3(j/2) \). It then follows that \( E_3(j/2) \cong \text{cok}(\delta_Y \delta_Z) \) for bidegree reasons. Similarly, there is no room for further differentials.

\[\square\]

Remark 4.6. The cyclic group \( \pi_{sk-1}(j) \) must map onto \( \pi_{sk-1}(j/2) \cong \mathbb{Z}/2 \) with a filtration shift of 1. The nontrivial \( \eta \)-multiplication on this latter group then implies the hidden \( \eta \)-extension from \( h_3 w^{k-1}_1 \) to \( c_3 w^{k-1}_1 \) for \( j \), as shown in Figure 6.

5. The odd-primary image-of-\( J \) spectrum

Let \( p \) be an odd prime, and let all spectra and homotopy groups be implicitly \( p \)-completed. The \( p \)-primary connective image-of-\( J \) spectrum \( j \) sits in a homotopy (co-)fiber sequence

\[
j \to \ell \xrightarrow{\psi} \Sigma^q \ell \to \Sigma j,
\]

where \( q = 2p - 2 \), \( r \) generates \( \mathbb{Z}^\times_p \) topologically, \( \ell \) is the Adams summand of connective complex \( K \)-theory, and \( \psi \) denotes a lift of \( \psi^j - 1 : \ell \to \ell \). Let \( E(1) = E[\beta, Q_1] \) and \( A(1) = \langle \beta, P^1 \rangle \) be the sub Hopf algebras of the mod \( p \) Steenrod algebra \( A \) generated by the listed elements, where \( Q_1 = P^1 \beta - \beta P^1 \). Let \( a \leq b \) mean that \( a \) is a \( p \)-adic unit times \( b \).

Proposition 5.1 ([Sin68]).

1. \( H^*(\ell) = A/A(\beta, Q_1) = A//E(1) \) and \( E_2(\ell) = \text{Ext}(E(1)(F_p, F_p) = F_p[v_0, v_1] \)

with algebra generators in \( (s, t) \)-bidegrees \( |v_0| = (1, 1) \) and \( |v_1| = (1, q + 1) \).

2. \( \pi_n(\ell) = \mathbb{Z}[v_1] \) with \( |v_1| = q \).

Proposition 5.2 ([Ada62]).

1. \( \psi^j(v_1) = r^{p-1}v_1 \).

2. \( \pi_0(j) = \mathbb{Z} \) and, when \( n \geq 1 \), \( \pi_n(j) = \begin{cases} \mathbb{Z}/pk & \text{for } n = kq - 1, \\ 0 & \text{otherwise}. \end{cases} \)

Proposition 5.3 ([Rog03, Prop. 5.1(b)]).

1. \( \psi^* : H^*(\Sigma^q \ell) \to H^*(\ell) \) is determined by \( \psi^* (\Sigma^q 1) \cong P^1 \).

2. \( K = \ker(\psi^*) = \Sigma^p A//A(1) \).

3. \( I = \text{im}(\psi^*) = \Sigma^p A/\langle \beta, Q_1, (P^1)^{p-1} \rangle \).

4. \( C = \text{cok}(\psi^*) = A//A(1) \).

In particular, \( K \cong \Sigma^p C \) as \( A \)-modules.

Proposition 5.4.

\[
\text{Ext}_A(C, F_p) = \text{Ext}_{A(1)}(F_p, F_p)
\]

\[
= F_p[v_0, b, w_1] \otimes E[a_i \mid 0 < i < p]/(\sim)
\]

is a free \( F_p[w_1] \)-module, where the ideal \( (\sim) \) imposes the relations \( v_0 a_i = 0 \) and

\[
a_i a_j = \begin{cases} (-1)^{i-1}v_0^{p-2}b & \text{for } i + j = p, \\ 0 & \text{otherwise}. \end{cases}
\]
The generators are graded as follows.

<table>
<thead>
<tr>
<th>s</th>
<th>$v_0$</th>
<th>$a_i$</th>
<th>b</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>1</td>
<td>$i(q + 1) - 1$</td>
<td>$pq$</td>
<td>$p(q + 1)$</td>
</tr>
<tr>
<td>$t - s$</td>
<td>0</td>
<td>$iq - 1$</td>
<td>$pq - 2$</td>
<td>$pq$</td>
</tr>
</tbody>
</table>

Proof. This is asserted without signs in [Hi08, Thm. 3.6]. We give a proof using the multiplicative Davis–Mahowald spectral sequence of [BR21, Ch. 2]. Let $P(0) = \langle P^1 \rangle = F_p[P^1]/((P^1)^p)$ be the sub Hopf algebra of $A(1)$ generated by $P^1$, so that $A(1)//P(0) = E[\delta, Q_1]$ as a left $A(1)$-module quotient coalgebra. Dually, $P(0)_c = F_p[\xi]/(\xi^p)$ is a quotient Hopf algebra of $A(1)_c = E[\tau_0, \tau_1] \otimes F_p[\xi]/(\xi^p)$, with $(A(1)//P(0))_c = E[\tau_0, \tau_1]$ as a left $A(1)_c$-comodule subalgebra. Let $R^* = F_p[v_1] \otimes E_p[w_1]$ be the graded $A(1)_c$-comodule algebra with coaction $\nu(v_0) = 1 \otimes v_0$ and $\nu(v_1) = 1 \otimes v_1 + \xi_0 \otimes v_0$, and give

$$(A(1)//P(0))_c \otimes R^*$$

the differential $\delta(\tau_0) = v_0$, $\delta(\tau_1) = v_1$, $\delta(v_0) = 0$, $\delta(v_1) = 0$ and the diagonal $A(1)_c$-coaction, making it a differentially graded $A(1)_c$-comodule algebra resolution of $F_p$.

We obtain an algebra spectral sequence $E_1^{s,t} = \text{Ext}_{A(1)}^{s,t}(F_p, F_p) \Rightarrow \text{Ext}_{A(1)}^*(F_p, F_p)$. Since $v_0^p \in R^p$ is $A(1)_c$-comodule primitive, there is an extension of algebra spectral sequences $F_p[v_0^p] \to E_1^{s,t} \to E_1^{s*,*}$ with $E_1^{s*,*} = \text{Ext}_{P(0)}^{s*,*}(R^p, \bar{R})$, where $\bar{R}^s = R^s/(v_1^p)$. Here $\bar{R}^{s-1} = F_p[v_0^{1-s}, \ldots, v_1^{1-s}]$ for $0 < s < p$, while $\bar{R}^0 = F_p[v_0^{1-s}, \ldots, v_1^{1-s}] \cong P(0)_c[v_0^{1-p}]$ for $i \geq p$. For $0 < i < p$ we have minimal injective resolutions of period 2

$$0 \to \bar{R}^{s-1} \xrightarrow{\phi_i} P(0)_c[v_0^{s-1}]/\Sigma^i P(0)_c[v_0^{s-1}] \xrightarrow{\delta^i} \Sigma^j P(0)_c[v_0^{s-1}] \rightarrow \ldots$$

with $\delta^i$ dual to multiplication by $(P^1)^i$ and $\delta^i$ dual to multiplication by $(P^1)^{p-i}$, which implies that

$$\text{Ext}_{P(0)}^{s*,*}(F_p, \bar{R}^s) = E_1^{s*,*}(F_p, F_p)$$

where $|a'_i| = (1, iq)$ and $|b| = (2, pq)$. For $i \geq p$, on the other hand, we have $\text{Ext}_{P(0)}^{s*,*}(F_p, \bar{R}^i) = F_p[v_0^{1-s}]$. The product $\phi: \bar{R}^1 \otimes \bar{R}^{p-1} \to \bar{R}^p$, which extends to a chain map $\phi_\sigma$ from the tensor product of the injective resolutions for $\bar{R}^1$ and $\bar{R}^{p-1}$ to the injective resolution for $\bar{R}^p$, in which

$$\phi_\sigma: \Sigma^\sigma P(0)_c[v_0^{s-1}] \otimes \Sigma^{p-q} P(0)_c[v_0^{s-1}] \to \Sigma^p P(0)_c[v_0^{s-1}]$$

maps $v_0^{s-1} \otimes v_0^{p-1}$ to $(-1)^{i-1}v_0^{p-1}$. The reader may prefer to verify this in the dual context of projective $P(0)$-module resolutions, keeping in mind that $\Sigma^p P(0)_c = (-1)^{i-1}$ mod. $p$.) It follows that

$$v_0^{i-1}a_i - v_0^{i-1}a_{p-i} = (-1)^{i-1}v_0^{p-2}b$$

in $E_1^{s*,*}$. On the other hand, the products $v_0 \cdot v_0^{i-1}a_i$ and $v_0^{i-1}a_i \cdot v_0^{j-1}a_{j-i}$ for $i + j \neq p$ are zero because they lie in trivial groups. The Davis–Mahowald spectral sequence collapses at the $E_1$-term, since all differentials on the algebra generators $v_0, v_0^{i-1}a_i$ and $v_0^{i-1}a_i$ land in trivial groups. Furthermore, there is no room for hidden multiplicative extensions. Letting $a \in \text{Ext}_{A(1)}^{q-1,i-1}(F_p, F_p)$, $b \in \text{Ext}_{A(1)}^{2pq}(F_p, F_p)$ and $w_1 \in \text{Ext}_{A(1)}^{2p}(F_p, F_p)$ be detected by $v_0^{i-1}a_i$, $b$ and $v_1^p$, respectively, we obtain the stated computation of $\text{Ext}_{A(1)}^{*}(F_p, F_p)$. 

$\square$
Lemma 5.5. The $A$-module extension
\[(E_i) \quad 0 \to C \overset{\partial}{\to} H^*(\ell) \overset{i}{\to} I \to 0 \]
duces a long exact sequence
\[\cdots \to E^{s,t}_2(\ell) \overset{j}{\to} \operatorname{Ext}^s_A(\ell, \mathbb{F}_p) \overset{\delta}{\to} \operatorname{Ext}^{s,t}_A(\ell, \mathbb{F}_p) \overset{g}{\to} E^{s,t}_2(\ell) \to \cdots \]
with
\[(\mathbb{F}_p[v_0, w_1] \{v_1, \ldots, v_1^{p-1}\} \cong \ker(\delta) \to \operatorname{Ext}^s_A(\ell, \mathbb{F}_p) \]
and
\[\operatorname{Ext}^{s,t}_A(\ell, \mathbb{F}_p) \to \cok(\delta) \cong \mathbb{F}_p[v_0, w_1].\]

For each nonzero $x \in \operatorname{Ext}^{s,t}_A(\ell, \mathbb{F}_p)$ that maps trivially to $\cok(\delta)$ there is a unique lift $\overline{x} \in \operatorname{Ext}^{s,t}_A(\ell, \mathbb{F}_p)$ satisfying $\delta(x) = x$, and $\operatorname{Ext}^s_A(\ell, \mathbb{F}_p)$ consists of these $\overline{x}$, extended by the free $\mathbb{F}_p[v_0, w_1]$-module on $i_y(v_1) = \nu_0\overline{x}_i$ for $0 < i < p$.

Lemma 5.6. The $A$-module extension
\[(E_{\Sigma|\ell}) \quad 0 \to I \overset{i}{\to} H^*(\Sigma^0\ell) \overset{i}{\to} K \to 0 \]
duces a long exact sequence
\[\cdots \to E^{s,t}_{2}(\Sigma^0\ell) \overset{j}{\to} \operatorname{Ext}^s_A(\Sigma^0\ell, \mathbb{F}_p) \overset{\delta}{\to} \operatorname{Ext}^{s,t}_A(\Sigma^0\ell, \mathbb{F}_p) \overset{g}{\to} E^{s,t}_{2}(\Sigma^0\ell) \to \cdots \]
with
\[(\Sigma^0\mathbb{F}_p[v_0, w_1] \{v_1, \ldots, v_1^{p-1}\} \cong \ker(\delta) \to \operatorname{Ext}^s_A(K, \mathbb{F}_p) \]
and
\[\operatorname{Ext}^{s,t}_A(K, \mathbb{F}_p) \to \cok(\delta) \cong \Sigma^0\mathbb{F}_p[v_0, w_1] \{v_1, \ldots, v_1^{p-1}\}.\]

For each nonzero $\overline{x} \in \operatorname{Ext}^{s,t}_A(K, \mathbb{F}_p)$ that maps trivially to $\cok(\delta)$ there is a unique lift $\overline{\overline{x}} \in \operatorname{Ext}^{s,t}_A(K, \mathbb{F}_p)$ satisfying $\delta(\overline{\overline{x}}) = \overline{x}$, and $\operatorname{Ext}^s_A(K, \mathbb{F}_p)$ consists of these $\overline{\overline{x}}$, extended by the free $\mathbb{F}_p[v_0, w_1]$-module on $i_y(v_1) = v_0^{-1}\overline{x}$.\n
Remark 5.7. $\operatorname{Ext}^s_A(K, \mathbb{F}_p)$ is a free $\operatorname{Ext}^s_A(\mathbb{F}_p, \mathbb{F}_p)$-module on one generator, $\overline{\overline{x}}$, as Proposition 5.3 indicates.

Lemma 5.8.
\begin{enumerate}
\item The kernel of $\delta : \operatorname{Ext}^s_A(K, \mathbb{F}_p) \to \operatorname{Ext}^{s+2}_A(C, \mathbb{F}_p)$ is
\[\ker(\delta) = \mathbb{F}_p[v_0, w_1] \{v_1^{p-1}\}.\]
\item The cokernel of $\delta : \operatorname{Ext}^{s-2}_A(K, \mathbb{F}_p) \to \operatorname{Ext}^s_A(C, \mathbb{F}_p)$ is
\[\cok(\delta) = \mathbb{F}_p[v_0, w_1] \oplus \mathbb{F}_p[w_1] \{a_1, \ldots, a_{p-1}\}.\]
\end{enumerate}

Proposition 5.9 ([Rog03, Prop. 5.1(b)]). The $A$-module extension
\[(E_j) \quad 0 \to C \overset{i}{\to} H^*(j) \overset{q}{\to} \Sigma^{-1}K \to 0 \]
is given by
\[H^*(j) = \frac{A(\{g_0, g_{pq-1}\})}{A(\{\beta(g_0), P^1(g_0), P^p(g_0) = \beta(g_{pq-1}), P^1(g_{pq-1})\})}\]
with $i(1) = g_0$ and $q(g_{pq-1}) = \Sigma^{pq-1}1$. In particular, it is induced up from an extension over $A(2) = \langle \beta, P^1, P^p \rangle$.\n
Proof. By change-of-rings, $\Ext^1_A(\Sigma^{-1} K, C) \cong \Ext^1_A(\Sigma^p q^{-1} F_p, A/A(1)) \cong F_p$. If the extension were trivial, then

$$E_2(j) \cong \Ext^1_A(F_p, F_p) \oplus \Ext^1_A(\Sigma^p q^{-1} F_p, F_p)$$

and the three classes $\Sigma^{-1} v_0^i b$ for $p-1 \leq i \leq p+1$ would survive to $E_\infty(j)$. This contradicts $\pi_{pq-1}(j) \cong \mathbb{Z}/p^2$. Hence the extension is nontrivial, which can only happen if $\beta(g_{pq-1})$ is a unit times $P^p(g_0)$.\hfill $\square$

**Lemma 5.10.** The $A$-module extension $(E_j)$ induces a long exact sequence

$$\cdots \to \Ext^s_A(\Sigma^{-1} K, F_p) \xrightarrow{\delta_X} \Ext^s_A(\Sigma^{-1} K, F_p) \xrightarrow{\delta_X} \Ext^s_A(C, F_p) \to \cdots$$

where $\delta_X$ is a derivation that vanishes on $v_0, a_i$ and $b$, with $\delta_X(v_1) \cong \Sigma^{-1} v_0^p b$. Hence $\delta_X$ maps

$$\text{coim}(\delta_X) = F_p[v_0] \{ w_1^k | k \neq 0 \mod p \}$$

isomorphically to

$$\text{im}(\delta_X) = F_p[v_0] \{ -v_0^{-1} w_1^{k-1} b | k \neq 0 \mod p \}.$$

**Definition 5.11.** Let $a_p \in E_2(j)$ be the image of $a_p^{-2} h_1$ in $E_2(S)$.

Up to a unit in $F_p$, this is also the image $q^*(\Sigma^{-1} v_0^{-1} \tilde{b}) = q^*(i_2^* (\Sigma^p v_0^{-1} b))$ of classes $\Sigma^{-1} v_0^{-1} \tilde{b} \in \Ext_A(\Sigma^{-1} K, F_p)$ and $\Sigma^p v_0^{-1} b \in \Ext_A(\Sigma^{-1} K, F_p)$. At these primes, $E_2(j)$ has Krull dimension 3, while $E_3(j)$ has Krull dimension 2.

**Theorem 5.12.** Let $j$ be the connective image-of-J spectrum at an odd prime $p$. There is an isomorphism

$$E_3(j) \cong F_p[w_1] \{ a_1, \ldots, a_{p-1} \}$$

$$\oplus F_p[w_1] \{ v_0^i a_p w_1^{k-1} | 0 \leq i \leq 1, 0 < k < p \}$$

$$\oplus F_p[w_1^2] \{ v_0^i a_p w_1^{k-1} | 0 \leq i \leq 2, 0 < k < p \}$$

$$\oplus F_p[w_0, w_1] \{ 1, a_p w_1^{k-1} \}$$

with generators in $(s, t)$-bidegrees $|v_0| = (1, 1)$, $|a_i| = (i, i(q+1) - 1)$ for $0 < i < p$, $|a_p| = (p-1, p(q+1) - 2)$ and $|w_1| = (p, p(q+1))$. The remaining nonzero differentials are

$$d_r(v_0^i w_1^k) = v_0^{r+1} a_p w_1^{k-1}$$

for $r \geq 3$, $i \geq 0$ and $\text{ord}_p(k) = r - 1$. Hence

$$E_\infty(j) \cong F_p[v_0]$$

$$\oplus F_p[w_1] \{ a_1, \ldots, a_{p-1} \}$$

$$\oplus \bigoplus_{r \geq 1} F_p[w_1^r] \{ v_0^i a_p w_1^{r-1-k+1} | 0 \leq i \leq r, 0 < k < p \}.$$

**Proof.** By Theorem 1.1, the differential $d_2 : E_2(j) \to E_2(j)$ maps the images under $q^*$ of the classes $\Sigma^{-1} \tilde{b}$ to lifts over $i^*$ of the classes $x$, mapping only the classes

$$F_p[\Sigma^{-1} v_0^{-1} b | 0 \leq i \leq 1 \text{ if } k \neq 0 \mod p]$$

of $\ker(\delta_2) / \text{im}(\delta_X)$ to zero. Lifts over $i^*$ of most classes $x$ become $d_2$-boundaries this way, leaving only the classes

$$F_p[v_0, w_1^p] \oplus F_p[w_1] \{ a_1, \ldots, a_{p-1} \}$$
of coker(\delta_1 \delta_2) \cap \ker(\delta_X). The direct sum of these two bigraded groups gives an upper bound for \(E_3(j)\), and comparing this with the known abutment, we must also have nonzero differentials
\[
d_2(w_1^{p^k}) = \Sigma^{-1}v_0^{p+2}w_1^{p^k-1}b
\]
for all \(k \not\equiv 0 \mod p\). Substituting \(a_p\) for the image under \(q^*\) of \(\Sigma^{-1}v_0^{p-1}b\), we obtain the stated Adams \(E_3\)-term for \(j\). The known abutment also determines the later differentials and the \(E_\infty\)-term. □

Remark 5.13. See Figure 8 for the case \(p = 3\). The map \(e: E_2(S) \to E_2(j)\) of mod \(p\) Adams spectral sequences detects some \(d_2\)-differentials in the domain, including those on \(h_1, g_2\) and \(u\) in the notation of [Nak75]. On the classes in the image of the \(J\)-homomorphism, the map \(e: \pi_{kq-1}(S) \to \pi_{kq-1}(j)\) preserves Adams filtration if \(\text{ord}_p(k) \in \{0, 1\}\), and otherwise increases Adams filtration by \(\text{ord}_p(k) - 1\).

Theorem 5.14. The Adams spectral sequence for \(j/p = j \wedge (S \cup_p e^1)\) with \(p\) odd has
\[
E_2(j) = E[a_1, \Sigma^{-1}b] \otimes \mathbb{F}_p[v_1, b]
\]
with \(d_2(\Sigma^{-1}b) = \pm b\) and
\[
E_3(j) = E_\infty(j) = E[a_1] \otimes \mathbb{F}_p[v_1]
\]
Here \(|a_1| = (1, q), |\Sigma^{-1}b| = (0, pq - 1), |v_1| = (1, q + 1)\) and \(|b| = (2, pq)\).

Proof. See [Rav86, Thm. 3.1.28] for \(\text{Ext}_{A(1)}(E[\beta], \mathbb{F}_p)\). □
Figure 4. \((E_2(j),\phi_2)\) at \(p = 2\)
Figure 5. Upper bound for $E_j$ with $d_2$ and $d_3$. 
The Adams Spectral Sequence for the Image-of-J Spectrum

Figure 6. $E_\infty$ with hidden $\eta$-extensions
Figure 7. $E_\infty(j/2) = E_\infty(j/2)$
Figure 8. $(E_2(j), d_2)$ at $p = 3$
References


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