Erich Ossa ([1]) computed the complex connective K-theory of elementary abelian groups by exploiting the idempotence of \( H^*BZ/(2) \) as a module over the exterior algebra \( E[Q_0, Q_1] \). At odd primes, the calculation splits into \((p - 1)^2\) pieces, each of which is simply a regrading of the result at \( p = 2 \).

Here, we use this method to give a precise calculation of \( ku_\ast(BC_p)^r \) focusing on odd primes \( p \). Let \( q = 2(p - 1) \).

A note on notation: we write \( ku \) for the spectrum formerly denoted \( bu \), reserving \( bu \) for the connected cover. Thus, \( ku \) has zeroth space \( BU \times \mathbb{Z} \) while \( bu \) has zeroth space \( BU \).

We have splittings

\[
ku \simeq l \vee \Sigma^2 l \vee \cdots \vee \Sigma^{2(p-2)} l
\]

where \( l \) is the Adams summand with mod \( p \) cohomology \( H^* l = A/E(1) \). Here \( E(1) = E[Q_0, Q_1] \), the exterior algebra on Milnor operations \( Q_0 \) and \( Q_1 \).

There is also a \( p \)-local splitting

\[
BC_p \simeq B_1 \vee B_2 \vee \cdots \vee B_{p-1}
\]

where \( B_i \) is a cell complex with cells in dimensions congruent to \( 2i - 1 \) and \( 2i \) modulo \( q \).

The Adams spectral sequence easily gives

\[
l_nB_1 = \begin{cases} \mathbb{Z}/(p^{j+1}) & n = jq + 1 \\ 0 & \text{otherwise} \end{cases}
\]

It follows that

\[
ku_nB_1 = \begin{cases} \mathbb{Z}/(p^{j+1}) & n = jq + 2k + 1, \ k = 0, \ldots, p - 2 \\ 0 & n \text{ even} \end{cases}
\]

Using the Thom isomorphism, Lemma 2.2.2 in [2] gives an equivalence \( ku \wedge B_{i+1} \simeq \Sigma^{2i} ku \wedge B_1 \), so we shall focus primarily on \( B_1 \).

Decomposing the \( E(1) \)-module \( H^* B_1 \otimes H^* B_1 \) into indecomposable \( E(1) \)-modules, we find, using Margolis’s theorem on \( H^* H \) summands, that

\[
l \wedge B_1 \wedge B_1 \simeq (\Sigma^2 l \wedge B_1) \vee \Sigma^2 HF_p[u_q, v_q]
\]

where \( u_q \) and \( v_q \) have degree \( q \). This is Proposition 4.2.3 in [2] with the details of the GEM there made precise, and comes from the characteristic free result, Lemma 4.2.1.

Assembling the summands of \( ku \), we find

\[
ku \wedge B_1 \wedge B_1 \simeq (\Sigma^2 ku \wedge B_1) \vee \Sigma^2 HF_p[u_2, v_q]
\]

where \( u_2 \) has degree 2.

Now summing the middle term in the smash product gives

\[
ku \wedge BC_p \wedge B_1 \simeq (\Sigma^2 ku \wedge BC_p) \vee \Sigma^2 HF_p[u_2, v_2]
\]

where \( v_2 \) also has degree 2.

Finally summing the third term in the smash product gives

\[
ku \wedge BC_p \wedge BC_p \simeq \bigvee_{i=1}^{p-1} ku \wedge BC_p \wedge B_1 \simeq \left( \bigvee_{i=1}^{p-1} ku \wedge \Sigma^2 BC_p \right) \vee \Sigma^2 HF_p[u_2, v_2, w_2]/(w_2^{p-1}).
\]
This is the result we iterate to describe \( ku \wedge (BC_p)^r \) in general. The simplest way to do the bookkeeping is to use the Poincare series of the cohomology before smashing with \( ku \). Since

\[
H^*(ku \wedge X) = A \otimes_{E_1} H^*X
\]

this amounts to writing the Poincare series of the \( E(1) \)-module which becomes the cohomology in question upon tensoring it up from \( E(1) \) to \( A \). Of course, this is simply the Poincare series of the cohomology divided by that of \( A \// E(1) \). Define

- \( R = (1 - t^i)/(1 - t^2) \), the Poincare series of \( F_p[u]/(u^{p-1}) \) where \( u \) has degree 2. This is the ratio of the Poincare series of \( H^*ku \) and \( H^*l \), or of \( H^*BC_p \) and \( H^*B_1 \).
- \( P = t/(1 - t) \), the Poincare series of \( H^*BC_p \).
- \( E_1 = (1 + t(1 + t^2)^{1/2}) \), the Poincare series of \( E(1) \).

The decomposition of \( ku \wedge BC_p \wedge BC_p \) gives the equation

\[
RP^2 = t^2 R^2 P + \frac{t^2 R}{(1 - t^2)^2} E_1
\]

The coefficient of \( E_1 \) is the Poincare series for the GEM summand. Since smashing with a GEM produces a GEM, we see by induction that

\[
RP^n = c_n R P + f_n E_1
\]

for some rational functions \( c_n \) and \( f_n \), and that this equation gives the corresponding decomposition of \( ku \wedge (BC_p)^{(r)} \). An easy induction provides \( c_{n+1} = (t^2 R)^n \) and

\[
f_{n+1} = \frac{(t^2 R)^n + (t^2 R)^{n-1} P + (t^2 R)^{n-2} P^2 + \ldots + (t^2 R)^{n-1} P^{n-1}}{(1 - t^2)^2} = \frac{1}{(1 - t^2)^2} \left( \frac{P^{n+1} - (t^2 R)^{n+1}}{P - t^2 R} - P^n \right).
\]

This gives

\[
ku \wedge (BC_p)^{(r+1)} \simeq \bigg( \bigvee_{i_1, \ldots, i_r=1}^{p-1} \Sigma^{2i_1} ku \wedge BC_p \bigg) \vee HM(f_{r+1})
\]

where \( i = i_1 + \cdots i_r \) and \( M(f) \) is the \( F_p \)-vector space with Poincare series \( f \). Reassembling the smash products to get the corresponding result for the Cartesian product, we find

\[
ku \wedge (BC_p)^n \simeq \bigg( \bigvee_{r=1}^{n} \binom{n}{r} \bigg( \bigvee_{i_1, \ldots, i_{r-1}=1}^{p-1} \Sigma^{2i_1} ku \wedge BC_p \bigg) \vee HM(f_{r}) \bigg)
\]

where \( i \) and \( M(\cdot) \) are as above.

When \( n = 3 \) this specializes to

\[
ku \wedge (BC_p)^3 \simeq 3(ku \wedge BC_p) \vee 3 \left( \bigvee_{i=1}^{p-1} \Sigma^{2i} ku \wedge BC_p \right) \vee HM(f_2) \vee \left( \bigvee_{i_1, i_2=1}^{p-1} \Sigma^{2(i_1 + i_2)} ku \wedge BC_p \right) \vee HM(f_3)
\]

To compare these results to [1], we first check that the \( ku \wedge BC_p \) summands work out correctly. The odd homotopy groups of \( ku \wedge BC_p \) have rank \( p - 1 \) after the first few, while the augmentation ideal in the representation ring of \( (C_p)^n \) has rank \( p^n - 1 \). Counting up the summands in (1) we see that we get the correct rank by the identity

\[
p^n - 1 = \sum_{r=1}^{n} \binom{n}{r} (p-1)^r.
\]

For the GEM summands, our results seem to agree for \( n < 3 \) and diverge thereafter. If \( n=p=3 \), the Poincare series for the GEM according to (1) is

\[
\frac{t^2(1+t^2)(3-2t+t^2-t^3+t^4+t^5)}{(1-t^2)^2(1-t)}
\]
while [1] gives
\[
\frac{t^2(1 + t^2)(3/2 + t/2 + t^2 + t^3/2 + t^4)}{(1 - t^2)^2}.
\]
The first discrepancy is in degree 7, where my Poincare series predicts the GEM summand to have rank 9, while Ossa’s predicts rank 8. The differences grow thereafter. I have checked that the Poincare series for the cohomology of the two sides of (1) agree out to degree 17 for \( p = 3, 5, 7, 11 \).

**REFERENCES**


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