Contemporary Mathematics
Volume **00**, 0000

Some remarks on the root invariant

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ABSTRACT. We show how the root invariant of a product depends upon the product of the root invariants, give some examples of the equivariant definition of the root invariant, and verify a weakened form of the algebraic Bredon-Löffler conjecture .

These remarks were worked out during the Stable Homotopy Theory Workshop at the Fields Institute in Toronto during January of 1996. The author would like to thank the organizers and the Fields Institute for support and for an environment conducive to doing mathematics.

1. The Definition and Some Examples

One of the most pleasing aspects of the equivariant point of view is the fact that concepts which are obscure non-equivariantly sometimes have quite clear equivariant meaning. Greenlees' observation [**3**] that Bredon's filtration of stable homotopy by restrictions of equivariant maps is the same as Mahowald's root invariant filtration is a good example, as it leads to an elementary definition of the root invariant as follows.

Let $G = Z/2$, let $\mathbb{R}^{n+k\xi}$ be the G-representation which is trivial on n coordinates and negation on k coordinates, and let $S^{n+k\xi}$ be the one point compactification of $\mathbb{R}^{n+k\xi}$. Let

$$\phi_k : [S^{k\xi}, S^0]_n^G \longrightarrow [S^0, S^0]_n = \pi_n S^0$$

and

$$U_k : [S^{k\xi}, S^0]_n^G \longrightarrow [S^k, S^0]_n = \pi_{n+k} S^0$$

be the fixed point and underlying map homomorphisms, respectively. Then [**3**, Prop. 2.5] shows that if k is maximal such that $x \in \pi_n S^0$ is in $\mathrm{Im}(\phi_k)$, then

$$R(x) = U_k(\phi_k^{-1}(x)).$$

1991 *Mathematics Subject Classification.* Primary 55Q45, 55Q35, 55Q91; Secondary 55P42, 55P91, 55T15 .

In other words, to compute $R(x)$, extend $x : S^n \longrightarrow S^0$ to an equivariant map $\widetilde{x} : S^{n+k\xi} \longrightarrow S^0$ with k maximal. Then $R(x)$ contains the underlying map $U_k(\widetilde{x}) : S^{n+k} \longrightarrow S^0$. Note that Lin's theorem, that $S^{-1} \simeq \varprojlim_k P_{-k}$, is not required for this definition.

The simplicity of this definition suggests that we should easily be able to see some examples. The Hopf maps bear this out. Let D be one of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or the Cayley numbers $\mathcal{C}$, and let $d_D = 1, 2, 4,$ or 8, respectively. The associated Hopf map $h_D : S^{2d-1} \longrightarrow DP^1 \cong S^d$ given by $h(z_1, z_2) = z_1 z_2^{-1}$ is 2, η, ν, or σ, respectively. Each of these division algebras is two-dimensional over the preceding one in the sequence. If we write such a pair as $D_1 \subset D_2$, then the elements of D_2 may be written $a + be$, with $a, b \in D_1$, $e \in D_2 - D_1$, and $e^2 = -1$. We may then give D_2 the G-action $a + be \mapsto a - be$, with respect to which h_{D_2} is then an equivariant map $S^{2d-1+2d\xi} \longrightarrow S^{d+d\xi}$ whose restriction to the fixed points is h_{D_1}. This shows that $R(2) = \eta$, $R(\eta) = \nu$, and $R(\nu) = \sigma$, once we verify that these extensions are maximal, which we will do in Corollary 3.

2. The Cartan Formula

The equivariant definition also allows an elementary proof of the Cartan formula, independent of the theory worked out in [**5**], which we present now.

Theorem 1. *Let $\alpha_i \in \pi_{n_i} S^0$ and $R(\alpha_i) \in \pi_{n_i+k_i} S^0$, for $i = 1, 2$. Let $k = k_1 + k_2$ and let $i : S^{-k-1} \longrightarrow P_{-k-1}$ be the inclusion of the bottom cell of the stunted projective space P_{-k-1}.*

(i) *If $i_*(R(\alpha_1)R(\alpha_2)) \neq 0$ then $R(\alpha_1)R(\alpha_2) \subset R(\alpha_1\alpha_2)$.*
(ii) *If $i_*(R(\alpha_1)R(\alpha_2)) = 0$ then $R(\alpha_1\alpha_2)$ lies in a higher stem than does $R(\alpha_1)R(\alpha_2)$.*

Proof: Let $\widetilde{\alpha_i} : S^{n_i+k_i\xi} \longrightarrow S^0$ be a maximal extension of α_i for each i. Then $\widetilde{\alpha} = \widetilde{\alpha_1} \wedge \widetilde{\alpha_2}$ is an extension of $\alpha = \alpha_1 \wedge \alpha_2$ and, if it is maximal then $R(\alpha_1\alpha_2)$ contains $U_k(\widetilde{\alpha}) = U_k(\widetilde{\alpha_1})U_k(\widetilde{\alpha_2}) = R(\alpha_1)R(\alpha_2)$. To determine whether or not it is maximal, we must analyze the relation between ϕ_k and ϕ_{k+1}.

Recall that the fixed point homomorphism $[S^{k\xi}, S^{\infty\xi}]^G_n \longrightarrow [S^0, S^0]_n$ is an isomorphism by elementary equivariant obstruction theory, so the inclusion of fixed points $i : S^0 \longrightarrow S^{\infty\xi}$ induces the fixed point homomorphism ϕ_k.

$$\begin{tikzcd}
{[S^{k\xi}, S^0]^G_n} \arrow[r, "i_*"] \arrow[dr, "\phi_k"'] & {[S^{k\xi}, S^{\infty\xi}]^G_n} \arrow[d, "\cong"] \\
& {[S^0, S^0]_n}
\end{tikzcd}$$

The G-equivariant cofiber sequence $EG_+ \longrightarrow S^0 \longrightarrow S^{\infty\xi}$ allows us to embed ϕ_k in a long exact sequence. Then, the cofiber sequence

$$S^k \wedge G_+ \longrightarrow S^{k\xi} \longrightarrow S^{(k+1)\xi},$$

allows us to compare ϕ_k and ϕ_{k+1}. Using the adjunction isomorphism $[S^k \wedge G_+, EG_+]^G_n \cong [S^k, EG_+]_n$, and the isomorphism $[S^k, EG_+]_n \cong [S^0, S^0]_n$ induced by the nonequivariant equivalence from EG_+ to S^0, we obtain the following braid of long exact sequences relating ϕ_k and ϕ_{k+1}. (This is a piece of the diagram used in [**3**] to show the equivalence of the two definitions of the root invariant.)

$$\begin{tikzcd}
 & {[S^{(k+1)\xi}, S^0]^G_n} \arrow[dr] \arrow[rr, bend left, "\phi_{k+1}"] & & {[S^0, S^0]_n} \arrow[dr] & \\
{[S^{(k+1)\xi}, EG_+]^G_n} \arrow[ur] \arrow[dr] & & {[S^{k\xi}, S^0]^G_n} \arrow[ur, "\phi_k"] \arrow[dr, "U_k"] & & {[S^{(k+1)\xi}, EG_+]^G_{n-1}} \\
 & {[S^{k\xi}, EG_+]^G_n} \arrow[ur] \arrow[rr, bend right] & & {[S^k, EG_+]_n} \arrow[ur, "\partial_k"] &
\end{tikzcd}$$

Commutativity of the right diamond shows that the obstruction to $\phi_k(\widetilde{\alpha})$ being in the image of ϕ_{k+1} is exactly $\partial_k U_k(\widetilde{\alpha})$. Thus, cases (i) and (ii) of the theorem are distinguished by the nontriviality or triviality, respectively, of the composite

$$S^{(k+1)\xi} \longrightarrow S^k \wedge G_+ \longrightarrow EG_+$$

of the boundary map of the cofiber sequence and the free G-map induced by $U_k(\widetilde{\alpha})$. To express this in more familiar terms, observe that naturality of Adams' isomorphism gives a commutative square

$$\begin{tikzcd}
{[S^k, EG_+]_n} \arrow[r, "\cong"] \arrow[d, "\partial_k"'] & {[S^k, S^0]_n} \arrow[r, "\cong"] & {[S^0, S^{-k}]_n} \arrow[d, "i_*"] \\
{[S^{(k+1)\xi}, EG_+]^G_{n-1}} \arrow[r, "\cong"] & {[S^0, EG_+ \wedge S^{-(k+1)\xi}]^G_{n-1}} \arrow[r, "\cong"] & {[S^0, P_{-k-1}]_{n-1}}
\end{tikzcd}$$

where $i : S^{-k} = \Sigma S^{-k-1} \longrightarrow \Sigma P_{-k-1}$ is inclusion of the bottom cell. This implies the form of the theorem we have stated. In fact, we have proved the following.

Proposition 2. *Suppose $\phi_k(\widetilde{\alpha}) = \alpha$. Then $\widetilde{\alpha}$ is maximal, and hence $R(\alpha) = U_k(\widetilde{\alpha})$, if and only if $0 \neq i_* U_k(\widetilde{\alpha}) \in \pi_{n-1} P_{-k-1}$.*

Corollary 3. *$R(2) = \eta$, $R(\eta) = \nu$, and $R(\nu) = \sigma$.*

Proof: Given the extensions of 2, η and ν produced in Section 1, we need only verify that η, ν, and σ are nontrivial on the bottom cells of P_{-2}, P_{-3}, and P_{-5}, a task which is easily accomplished.

Previous versions of the Cartan formula have appeared in [**9, 12**], with the condition for $R(\alpha_1)R(\alpha_2)$ to be contained in $R(\alpha_1\alpha_2)$ stated as $R(\alpha_1)R(\alpha_2) \neq 0$ rather than $i_*(R(\alpha_1)R(\alpha_2)) \neq 0$.

Here is an example which can be found in [**10**]. If $\mu \in \pi_8 S$ is detected by Ph_1, and $\bar{\kappa} \in \pi_{20} S$ detected by $g \in \mathrm{Ext}^{4,24}$ is chosen correctly then we have

$R(\mu) = \nu\bar{\kappa}$. To compute $R(\eta\mu)$ we thus consider the product $R(\eta)R(\mu) = \nu^2\bar{\kappa}$ This is nonzero in $\pi_{26}S = \pi_9 S^{-17}$. However $i_*(\nu^2\bar{\kappa}) = 0$ in $\pi_9 P_{-17}$, so we conclude that $R(\eta\mu)$ lies in a stem higher than π_{26}.

The algebraic root invariant, discussed in the next section, exhibits the same behavior. There, we have $R(h_1) = h_2$, $R(Ph_1) = h_2 g$, and $R(h_1)R(Ph_1) = h_2^2 g \neq 0$ in $\mathrm{Ext}(F_2, F_2)$. However,

$$i_* : \mathrm{Ext}(F_2, F_2) \longrightarrow \mathrm{Ext}(H^*P_{-17}, F_2)$$

sends $h_2^2 g$ to zero, so the algebraic root invariant $R(h_1 Ph_1)$ lies in a higher stem as well. In fact, we can calculate that $R(h_1 Ph_1) = r$, which lies in the 30-stem and "lives on" the bottom cell of P_{-21}. These calculations were done by computing the induced maps of Adams spectral sequences for the spectra involved, using the programs described in [**1**] and [**2**]. It was the anomolous behavior of $R(h_1 Ph_1)$ in those calculations which alerted me to the correct formulation of the Cartan formula.

In terms of the root invariant spectral sequence of [**5**], the 'exceptional' behavior in case (ii) of the theorem is the usual behavior of products in an associated graded when there are filtration shifts.

The condition for maximality allows us to make systematic conclusions, along the lines of [**11**], generalizing the example above. Let us write $|x| = n$ if $x \in \pi_n S^0$.

Corollary 4. *For any* $x \in \pi_* S^0$,

(i) $|R(2x)| > 1 + |R(x)|$ *if* $|R(x)| - |x| \equiv -1$ *(mod 4),*
(ii) $|R(\eta x)| > 3 + |R(x)|$ *if* $|R(x)| - |x| \equiv -2$ *(mod 8),*
(iii) $|R(\nu x)| > 7 + |R(x)|$ *if* $|R(x)| - |x| \equiv -4$ *(mod 16).*

Proof: The map η is trivial on the bottom cell of P_{-j-2} if $-j-1 \equiv -1$ (mod 4). Similarly, ν is trivial on the bottom cell of P_{-j-3} if $-j-3 \equiv -1$ (mod 8), and σ is trivial on the bottom cell of P_{-j-5} if $-j-5 \equiv -1$ (mod 16).

3. The Algebraic Bredon-Löffler Conjecture

As noted in [**3**], the equivariant definition of the root invariant also allows a very simple proof of Jones' [**6**] lower bound

$$|R(x)| \geq 2|x|.$$

Namely, any $x : S^n \longrightarrow S^0$ occurs as the fixed points of $x \wedge x : S^{n+n\xi} = S^n \wedge S^n \longrightarrow S^0 \wedge S^0 = S^0$, where the smash products are given the action which interchanges factors. The *Bredon-Löffler Conjecture* is the upper bound

$$|R(x)| \leq 3|x|$$

when $|x| > 0$. By [**3**], this is equivalent to the assertion that the map $\eta_k : S^0 \longrightarrow \Sigma P_{-k}$ induces a monomorphism of π_j for $0 < j < k/2$. The Adams spectral sequence gives an algebraic analog, which we call the *Algebraic Bredon-Löffler Conjecture*, namely, that

$$\eta_k^* : \operatorname{Ext}_A^{s,t}(F_2, F_2) \longrightarrow \operatorname{Ext}_A^{s,t}(H^*\Sigma P_{-k}, F_2)$$

is a monomorphism for $0 < t-s < k/2$. We are able to prove the following much weaker assertion.

THEOREM 5. *η_k^* is a monomorphism for $0 < t - s < -2 + \sqrt{3 + k/2}$.*

Proof: Let ρ_r be the natural transformation $\operatorname{Ext}_A \longrightarrow \operatorname{Ext}_{A_r}$ induced by the inclusion $A_r \longrightarrow A$ of the subalgebra generated by $Sq^1, \dots, Sq^{2^r}$. We will show that $\rho_r \eta_k^*$ is a monomorphism of $\operatorname{Ext}_A^{s,t}$ in the stated range if r is minimal such that $n = t - s < 2^r - 1$. We shall use several facts from [**8**]. Let $L = F_2[x, x^{-1}]$ and let $j_k : L_{-k} = H^*P_{-k} \longrightarrow L$ be the obvious inclusion. Let $\lambda : \Sigma L \longrightarrow F_2$ be the coefficient of x^{-1}. It is easy to see that $\lambda j_k = \eta_k^* : \Sigma L_{-k} \longrightarrow F_2$. Further, we have:

(i) The induced homomorphism $\lambda^* : \operatorname{Ext}_A(F_2, F_2) \longrightarrow \operatorname{Ext}_A(\Sigma L, F_2)$ is an isomorphism [**8**, Theorem 1.1].

(ii) There is an isomorphism

$$\operatorname{Ext}_{A_r}(\Sigma L, F_2) \cong \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_{A_{r-1}}(\Sigma^{j2^{r+1}} F_2, F_2).$$

See [**8**, Theorem 1.1] and [**4**, Theorem 2.1].

(iii) The lower left square in the following diagram commutes. That is, the preceding isomorphism composed with λ_* and projected onto the zeroth component, is induced by the inclusion $A_{r-1} \longrightarrow A_r$ [**8**, Lemma 1.6].

$$\begin{tikzcd}
\operatorname{Ext}_A(F_2,F_2) \arrow[r, "\cong", "\lambda^*"'] \arrow[d, "\rho_r"'] \arrow[rr, bend left, "\eta_k^*"] \arrow[dd, bend right=60, "\rho_{r-1}"'] & \operatorname{Ext}_A(\Sigma L,F_2) \arrow[r, "j_k^*"'] \arrow[d, "\rho_r"'] & \operatorname{Ext}_A(\Sigma L_{-k},F_2) \arrow[d, "\rho_r"'] \\
\operatorname{Ext}_{A_r}(F_2,F_2) \arrow[r, "\lambda^*"'] \arrow[d] & \operatorname{Ext}_{A_r}(\Sigma L,F_2) \arrow[r, "j_k^*"'] \arrow[d, "\cong"'] & \operatorname{Ext}_{A_r}(\Sigma L_{-k},F_2) \\
\operatorname{Ext}_{A_{r-1}}(F_2,F_2) & \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}_{A_{r-1}}(\Sigma^{j2^{r+1}}F_2,F_2) \arrow[l, "\pi_0"'] &
\end{tikzcd}$$

The left hand map ρ_{r-1} is a monomorphism on $\operatorname{Ext}_A^{s,t}(F_2, F_2)$ since $n = t - s < 2^r - 1$. Hence $\rho_r \lambda^*$ is as well. Now Adams' vanishing line implies that $\operatorname{Ext}_A^{s,t}(F_2, F_2) = 0$ unless $s < (n + 4)/2$, so we will be done if we can show that the lower map j_k^* is a monomorphism in these filtrations. This will be true if $s < (n + k)/(2^{r+1} - 2)$, because then $\operatorname{Ext}_{A_r}^{s,t}(\Sigma(L/L_{-k}), F_2) = 0$. This follows by filtering L/L_{-k} by degrees and using the fact that $\operatorname{Ext}_{A_r}^{s,t}(F_2, F_2) = 0$

if $s < n/(2^{r+1} - 2)$ by the May spectral sequence, c.f. [**4**, proof of Theorem 2.1]. Thus, it suffices to have

$$\frac{n+4}{2} < \frac{n+k}{2^{r+1}-2}.$$

Since r is minimal with respect to $n < 2^r - 1$, we have $2^r - 1 \leq 2n$, so it suffices that $2n^2 + 7n < k$, which holds if $n < -2 + \sqrt{3 + k/2}$.

Of course this is far from the actual conjecture, but is a first step toward it. In fact, the calculations in [**3**] suggest the following sharpening of the algebraic Bredon-Löffler conjecture.

Conjecture 6. *(Strong Algebraic Bredon-Löffler Conjecture) The map η_k^* is a monomorphism if*

$$s < (k - n)/2.$$

This conjecture is based on calculations of root invariants [**3**]. The sparsity of elements in some bidegrees introduces an element of uncertainty in the intercept, but the slope of $-1/2$ and the approximate intercept $n = k$ are clearly evident for $n < 40$.

Adams' vanishing line intersects this line at about $n = k/2$, so this conjecture implies the algebraic Bredon-Löffler conjecture (see Figure 3). Since $k/2$ is the last entire stem which is mapped monomorphically, this is the dimension one sees in homotopy.

This conjecture also correctly predicts that part of the 0-stem which maps monomorphically according to Landweber [**7**], unlike the algebraic Bredon-Löffler conjecture which says nothing about the 0-stem.

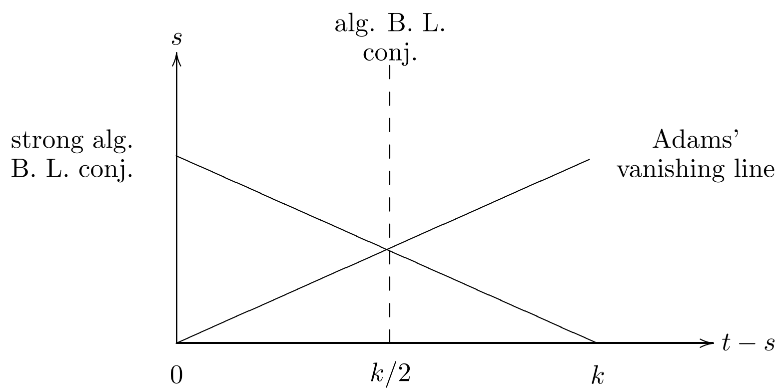

Figure 1. Conjectured limits to the range in which η_k^* is a monomorphism.

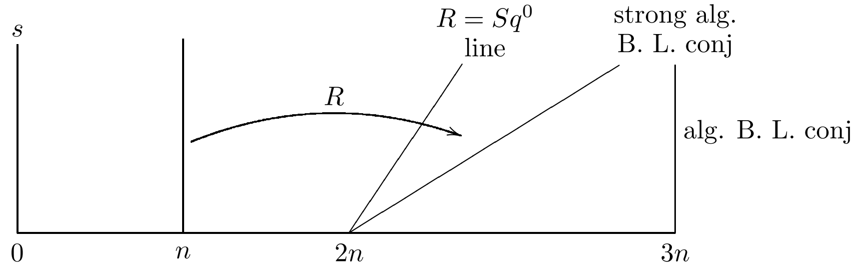

Figure 2. Root invariants of the n-stem

Finally, note that this confines root invariants to a narrow band,

$$2|x| + s \ \leq\ |R(x)| \ \leq\ 2|x| + 2s,$$

as in Figure 3. Here, the lower bound follows from the result of [**9**, 2.5], that $R(x) = Sq^0(x)$ (*if* the latter is nonzero on the appropriate cell of projective space), and the upper bound is equivalent to the strong algebraic Bredon-Löffler conjecture.

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