ASYMMETRY AND EFFICIENCY IN TODA BRACKETS

ROBERT R. BRUNER

Abstract. We show that in the category of chain complexes, Toda brackets can be computed using far less information than is commonly supposed. This has proven useful in automating homological algebra.

Let k be a commutative ring with unit and let A be an augmented k-algebra with augmentation ideal I. We will work in the category of chain complexes of A-modules. For the purposes of this note we will further assume that k has characteristic 2 in order to avoid sorting out the correct signs.

Given chain maps $C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \xrightarrow{h} F_*$ and chain null-homotopies $\eta : gf \simeq 0$ and $\mu : hg \simeq 0$, we can construct a new chain map

$$T = h\eta + \mu f : C_* \longrightarrow F_*$$

called *a Toda bracket* by analogy with the construction of this name in the homotopy theory of spaces. Strictly speaking, *the Toda bracket* is the set obtained by letting η and μ vary over all nullhomotopies:

$$\langle h, g, f \rangle = \{ h\eta + \mu f : \eta : gf \simeq 0, \mu : hg \simeq 0 \}$$

and it is standard that, under the hypotheses below, if $T \in \langle h, g, f \rangle$, then

$$\langle h, g, f \rangle = T + h[C_*, E_*] + [D_*, F_*]f$$

where $[-,-]$ denotes the set of chain maps. (This follows from the fact that two null-homotopies differ by a chain map. This is one place signs must be handled carefully in characteristic not 2.)

Our attitude will be that we are only required to produce one element of any Toda bracket, and the general theory is responsible for telling us the remainder of the set.

The point of this note is to show that if the complexes C_* and D_* are projective, if the complex D_* is minimal (i.e., $d(D_i) \subset I \cdot D_{i-1}$ for all i) and if F_* is a resolution of k, then the Toda bracket $T = h\eta + \mu f$ is completely determined by f, g, h and η. That is, we can produce an element T of the Toda bracket without using μ, and as noted, all other elements of the bracket can then be obtained using only f, h and T. The most common application is to the case where $C_* = D_* = E_* = F_*$ is a minimal resolution of k by projective A-modules. Dave Benson has suggested that this is probably a result about A_∞-algebras.

Let us fix notations. The chain maps f, g and h have components

$$\begin{aligned} f_i &: C_{p+i} \longrightarrow D_i \\ g_i &: D_{q+i} \longrightarrow E_i \\ h_i &: E_{r+i} \longrightarrow F_i \end{aligned}$$

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We write d generically for the differential of C_*, D_*, E_* or F_*. The null homotopies η and μ have components

$$\eta_i : C_{p+q+i-1} \longrightarrow E_i$$
$$\mu_i : D_{r+r+i-1} \longrightarrow F_i$$

and satisfy

$$\begin{aligned} g_i f_{q+i} &= \eta_i d + d\eta_{i+1} \\ h_i g_{r+i} &= \mu_i d + d\mu_{i+1} \end{aligned}$$

Our hypotheses on C_*, D_* and F_* imply that chain homotopy classes of chain maps $C_* \longrightarrow F_*$ or $D_* \longrightarrow F_*$ are in one-to-one correspondence with cohomology classes of cocycles $C_* \longrightarrow k$ or $D_* \longrightarrow k$, respectively, sending a chain map $x : D_* \longrightarrow F_*$, for example, to the cocycle $\varepsilon x_0 : D_i \longrightarrow F_0 \longrightarrow k$, where $\varepsilon : F_0 \longrightarrow k$ is the cokernel of $d : F_1 \longrightarrow F_0$.

The crucial point is that we may assume that $\mu_0 = 0$. We see this as follows. Given a null-homotopy $\nu : hg \simeq 0$ we have $h_0 g_r = \nu_0 d + d\nu_1$. The cocycle corresponding to hg is $\varepsilon h_0 g_r = \varepsilon\nu_0 d + \varepsilon d\nu_1$. Now $\varepsilon d = 0$ always, and $\varepsilon\nu_0 d = 0$ by minimality:

$$\varepsilon\nu_0 d(D_{q+r}) \subset \varepsilon\nu_0(I \cdot D_{q+r-1}) \subset I \cdot k = 0$$

so $\varepsilon h_0 g_r = 0$. Thus, $h_0 g_r$ factors as $d\mu_1$ and we may construct a new null-homotopy $\mu : hg \simeq 0$ in which $\mu_0 = 0$.

This implies that

$$T_0 = h_0\eta_r + \mu_0 f_{q+r-1} = h_0\eta_r$$

so the cocycle corresponding to T is $\varepsilon T_0 = \varepsilon h_0 \eta_r$. This cocycle determines T up to chain homotopy, and does not depend upon μ.

It may be reassuring to check that $\varepsilon h_0\eta_r$ is a cocycle:

$$\begin{aligned} \varepsilon h_0 \eta_r d &= \varepsilon h_0(g_r f_{q+r} - d\eta_{r+1}) \\ &= \varepsilon h_0 g_r f_{q+r} - \varepsilon h_0 d\eta_{r+1} \\ &= 0 - \varepsilon d h_1 \eta_{r+1} \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

□

Final note: Clearly the same argument applies if F_* is a resolution of M and $d(D_i) \subset Ann(M) \cdot D_{i-1}$ for each i.

These methods extend to longer Toda brackets as well. Here is the length four case. Given chain maps

$$B \xrightarrow{a_3} C \xrightarrow{a_2} D \xrightarrow{a_1} E \xrightarrow{a_0} F$$

with null homotopies a_{ij} satisfying

$$\begin{aligned} da_{01} + a_{01}d &= a_0a_1 \\ da_{12} + a_{12}d &= a_1a_2 \\ da_{23} + a_{23}d &= a_2a_3 \\ da_{02} + a_{02}d &= a_0a_{12} + a_{01}a_2 \\ da_{13} + a_{13}d &= a_1a_{23} + a_{12}a_3 \end{aligned}$$

showing that the composites a_ia_{i+1} and the Toda brackets $\langle a_i, a_{i+1}, a_{i+2}\rangle$ all contain 0, we have the four fold Toda bracket

$$\langle a_0, a_1, a_2, a_3\rangle := a_0a_{13} + a_{01}a_{23} + a_{02}a_3,$$

or rather, the Toda bracket is the set of all such as the null-homotopies are varied.

Just as above, if the complexes are minimal and F is a resolution of the ground ring, or more generally if the images of the appropriate differentials lie in the annihilator ideal of the the module resolved by F, then we can find an element of this four fold bracket by computing a_0a_{13} only, or ore economically, the cocycle it represents, which means that we really only need the null homotopy a_{13} and the cocycle corresponding to a_0.

A further reduction is possible when a_0 has homological degree 1. Just as the chain map litfing a cocycle a_2 allows one to compute all Toda brackets $\langle a_0, a_1, a_2\rangle$, the null homotopy a_{23} of a_2a_3 allows one to compute all Toda brackets $\langle a_0, a_1, a_2, a_3\rangle$.

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References

1. R. R. Bruner, "Calculation of Large Ext Modules", Ch. 4 in *Computers in Geometry and Topology,* Martin C. Tangora Ed., Marcel Dekker, Inc., (1989), 79-104.

Universitetet i Oslo, Department of Mathematics, Postboks 1053, Blindern, 0316 Oslo, Norway
E-mail address: rrb@math.uio.no

Mathematics Department, Wayne State University, Detroit, Michigan, 48202, USA
E-mail address: rrb@math.wayne.edu